September 30, 1964 R-AERO-IN-23-64

# INTERNAL NOTE

Aero-Astrodynamics Internal Note #23-64

## A COMPARISON OF AN MIT EXPLICIT GUIDANCE PRINCIPLE WITH MSFC ITERATIVE GUIDANCE


By

Judson J. Hart

DYNAMICS AND FLIGHT MECHANICS DIVISION
AERO-ASTRODYNAMICS LABORATORY

GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama


FOR INTERNAL USE ONLY

GEORGE C. MARSHALL SPACE FLIGHT CENTER

Aero-Astrodynamics Internal Note 23-64

September 30, 1964

# A COMPARISON OF AN MIT EXPLICIT GUIDANCE PRINCIPLE WITH MSFC ITERATIVE GUIDANCE

By

Judson J. Hart



DYNAMICS AND FLIGHT MECHANICS DIVISION
AERO-ASTRODYNAMICS LABORATORY

## DEFINITION OF SYMBOLS

| Symbol | Definition |
|---|---|
| $r$ | distance from the center of the spherical attracting body |
| $\mu$ | $ge\ re^2$ |
| $re$ | earth radius |
| $ge$ | sea level gravitation |
| $a_T$ | thrust acceleration |
| $V_\theta$ | velocity in the local horizontal direction |
| $\alpha$ | direction of the thrust vector against the local horizontal |
| $V_e$ | $ge_{I_{sp}}$ exhaust velocity |
| $\tau$ | $\frac{m_o}{\dot{m}} = \frac{V_e}{a_T}$; burnup time |
| $T$ | time at which desired end conditions are met |
| $T_{go}$ | time from current state to satisfaction of end conditions |

Subscripts

| | |
|---|---|
| D | values at desired end condition |
| 0 | values at current state |

Aero-Astrodynamics Internal Note 23-64

# A COMPARISON OF AN MIT EXPLICIT GUIDANCE PRINCIPLE WITH MSFC ITERATIVE GUIDANCE


## SUMMARY

Although they are not precisely equivalent, the MIT guidance principle and the MSFC iterative guidance have many similarities. Both schemes steer toward a specified end point. The MIT scheme uses thrust to cancel out the effective gravity, a nonlinear term, which may be inefficient in certain cases. The MSFC scheme is more closely connected with calculus of variations and optimization theory in a reasonable degree of approximation. For many missions, the performance differences are probably insignificant. The MIT scheme is difficult to generalize to multiple stage operation whereas the MSFC scheme has demonstrated its effectiveness in multi-stage applications numerous times. In fact, the MSFC scheme has demonstrated its capability in all the situations for which the MIT guidance scheme is proposed. The MIT scheme is simple and economical regarding in-flight computer requirements and has merit where guidance over only a single stage is required. Some modification of the time-to-go prediction is desirable to eliminate an iteration process in flight. Detailed numerical studies on the MIT scheme will be reported on when they are available. Actual differences in computer requirements between single stage MSFC iterative guidance and MIT guidance are not significant.


## I. INTRODUCTION

A great deal of work has been done in this country since 1961 in the area of explicit guidance for space vehicles. MIT has made certain studies in this field for use in Service Module and LEM applications. MSFC has also worked in the explicit guidance field during these years, the results being known as iterative guidance, a guidance mechanization which has been exceedingly useful in ascent, lunar transit, and lunar landing applications. MSFC iterative guidance has arisen from trajectory studies in calculus of variations and from the need for a simple explicit guidance formulation which would be adaptable to all possible perturbations and be capable of a high payload performance. The purpose of this report is to explore the principles underlying MIT guidance assumptions and to compare them with the assumptions and results of the theoretical optimum obtained by calculus of variations. Some comparison is made also with regard to ease of implementation.

## II. PITCH PLANE GUIDANCE PRINCIPLE

Equations of motion formulated by MIT in a polar coordinate system are

$$\ddot{r} = -\frac{\mu}{r^2} + \frac{V_\theta^2}{r} + a_T \sin \alpha \tag{1a}$$

$$\dot{V}_\theta = -\frac{\dot{r}V_\theta}{r} + a_T \cos \alpha. \tag{1b}$$

The thrust acceleration is defined as $a_T$ in direction $\alpha$ against the local horizontal, $V_\theta$ is the component of velocity in the local horizontal direction, and r is the distance from the spherical attracting body. Values for

$$\dot{r}_o = \dot{r}(t_o) \tag{2a}$$

$$r_o = r(t_o) \tag{2b}$$

$$V_{\theta o} = V_\theta(t_o) \tag{2c}$$

at any instant of time, $t_o$, are provided by the inertial measuring unit after some transformation. Desired end conditions,

$$\dot{r}(T) = \dot{r}_o \tag{3a}$$ ? $\dot{r}_D$ ?

$$r(T) = r_D, \tag{3b}$$

are those specified by the mission objective, i.e., the attainment of a circular orbit at some specified altitude as an example. The time T is that time when the satisfaction of these desired end conditions takes place. The problem is to find $\alpha(t)$ in a manner permitting satisfaction of the end conditions and permitting an efficient use of fuel.

Now

$$\dot{r}(t) = \dot{r}_o + \int_{t_o}^{t} \ddot{r}(s)\ ds \tag{4}$$

$$\dot{r}_o - \dot{r}_o = \int_{t_o}^{T} \ddot{r}(t)\ dt \tag{5}$$

$$r_D - r_o - \dot{r}_o\ T_{go} = \int_{t_o}^{T} \left[ \int_{o}^{t} \ddot{r}(s)\ ds \right] dt \tag{6}$$

where

$$T_{go} = T - t_o \tag{7}$$

and where $\ddot{r}(t)$ is constrained by the necessity to meet the desired end conditions in an efficient way. It is clear that $\ddot{r}$ needs two degrees of freedom to satisfy the two end conditions $r(T)$ and $\dot{r}(T)$.

Assume that

$$\ddot{r}(t) = Ap(t) + Bq(t) \tag{8}$$

where optimization considerations can be used in the selection of $p(t)$ and $q(t)$ and end conditions determined by the choice of A and B.

Integrating equation (7) gives

$$\dot{r}_D - \dot{r}_o = A \int_{t_o}^{T} p(t)\ dt + B \int_{t_o}^{T} q(t)\ dt \tag{9}$$

$$\dot{r}_D - r_o - \dot{r}_o\ T_{go} = A \int_{t_o}^{T} \left[ \int_{t_o}^{t} p(s)\ ds \right] dt + B \int_{t_o}^{T} \left[ \int_{t_o}^{t} q(s)\ ds \right] dt. \tag{10}$$

Thus,

$$\dot{r}_D - \dot{r}_o = f_{11}A + f_{12}B \tag{11}$$

$$r_D - r_o - \dot{r}_o\, T_{go} = f_{21}A + f_{22}B \tag{12}$$

where the coefficients $f_{ij}$ are defined by the appropriate integrals in equations (9) and (10) and are functions of $T_{go}$. Assuming the $f_{ij}$ and $T_{go}$ are known, A and B may be found in a linear fashion.

$$A = e_{11}\,[\dot{r}_D - \dot{r}_o] + e_{12}\,[r_D - (r_o + \dot{r}_o\, T_{go})] \tag{13a}$$

$$B = e_{21}\,[\dot{r}_D - \dot{r}_o] + e_{22}\,[r_D - (r_o + \dot{r}_o\, T_{go})] \tag{13b}$$

with

$$e_{11} = \frac{f_{22}}{\Delta}; \quad e_{12} = -\frac{f_{12}}{\Delta}; \quad e_{21} = -\frac{f_{21}}{\Delta}; \quad e_{22} = \frac{f_{11}}{\Delta}$$

and

$$\Delta = f_{11}\, f_{12} - f_{12}\, f_{21}.$$

The reason for the assumption of equation (8) is apparent: It permits equations (9) and (10) to be solved for A and B in a linear fashion, if $T_{go}$ is known. The prediction of $T_{go}$ will be discussed later.

Once A and B are computed, the thrust direction $\alpha$ may be computed from the combination of equations (8) and (1a) solved for $\alpha$.

$$\sin\alpha = \frac{1}{a_T}\left\{Ap(t) + Bq(t) - \left[\frac{\mu}{r^2} + \frac{V_\theta^2}{r}\right]\right\} \tag{14}$$

where $a_T$, r, and $V_\theta$ are obtained from the navigation system. The computation is recycled as time progresses to generate new attitude commands until $T_{go}$ becomes small, since as $T_{go} \to 0$ the determinant $\Delta \to 0$. The MIT scheme provides only thrust angle, but not angular rate, as does the iterative guidance scheme. Also the MIT principle requires the arc sine to be evaluated to get the thrust direction. Finally $\alpha$ must be transformed into the platform coordinate system.

## III. SPECIFIC GUIDANCE SCHEMES

Several distinct guidance schemes have arisen from MIT work depending on the assumed form of p(t) and q(t) in equation (8). The simplest scheme and the one proposed for certain LEM operations assumes

$$p(t) = 1; \; q(t) = T - t.$$

Hence,

$$\ddot{r} = A + B(T - t) \tag{15}$$

is assumed. This assumption cannot be justified on any basis other than simplicity. The assumption that $\ddot{r}$ is linear over time is not justified from the standpoint of optimization. The function $\ddot{r}$, taken from a typical Saturn IB second stage trajectory to 105 n. mi. orbit and computed using calculus of variations, is shown in Figure 1. It is conceivable that a scheme based on this primitive assumption can lead to reasonably good payload performance since A and B are repetitively evaluated and not held constant over flight time. Hence, although linearity is assumed, it is not forced.

This specific scheme has been called E* guidance and results in exceedingly simple guidance equations.

$$f_{11} = T_{go}; \quad f_{12} = \frac{T_{go}^2}{2}; \; f_{21} = \frac{T_{go}^2}{2}; \quad f_{22} = \frac{T_{go}^3}{3}; \; \Delta = \frac{1}{12} T_{go}^4 \tag{16}$$

$$a_{11} = - V_e \ln \left(1 - \frac{T_{go}}{\tau}\right) \qquad (21a)$$

$a_{1j}$ correspond to $f_{ij}$ of previous E* scheme.

$$a_{12} = a_{11} \tau - V_e T_{go} \qquad (21b)$$

$$a_{21} = - a_{12} + a_{11} T_{go} \qquad (21c)$$

$$a_{22} = a_{21} \tau - \frac{1}{2} V_e T_{go}^2 \qquad (21d)$$

$$\Delta = a_{11} a_{22} - a_{12} a_{21} \qquad (21e)$$

$$b_{11} = \frac{a_{22}}{\Delta};\ b_{12} = - \frac{a_{12}}{\Delta};\ b_{21} = - \frac{a_{21}}{\Delta};\ b_{22} = \frac{a_{11}}{\Delta} \qquad (21f)$$

$$C = b_{11} [\dot{r}_D - \dot{r}_o] + b_{12} [r_D - (r_o + \dot{r}_o T_{go})] \qquad (22a)$$

$$D = b_{21} [\dot{r}_D - \dot{r}_o] + b_{22} [r_D - (r_D + \dot{r}_o T_{go})] \qquad (22b)$$

$$\sin \alpha = - \frac{1}{a_T} \left[ - \frac{\mu}{r^2} + \frac{V_\theta^2}{r} \right] + C + Dt \qquad (23)$$

where r, $V_\theta$ at the current state are used and t is measured from the last computation of C and D. MIT has stated that this version of the guidance scheme has given better performance results than that obtained with a steepest descent trajectory optimization program.

MIT has evaluated schemes in between the assumptions of equations (15) and (20). For example, in equation (20) one may assume

$$\ddot{r} = Ca_T + Da_T (T - t)$$

and expand

$$A_T = \frac{V_e}{\tau - t}$$

in a Taylor series of two or three terms in powers of T - t. In this way, the logarithm in equations (21) is eliminated and the corresponding coefficients to equations (21) or (16) become polynomials in $T_{go}$. Details of this approach are not presented, for brevity, and the series assumed is poorly convergent. There are several versions of the basic MIT guidance principle presented here as far as implementation is concerned, all of which are similar in that an assumption primarily for convenience is made which allows the $\ddot{r}$ differential equation to be linearized and treated in closed form, so that the boundary value problem can be solved. The approach toward MSFC iterative guidance is directly suggested by calculus of variations and is not equivalent to the assumptions of MIT. For example, the assumption about thrust direction in equation (18) cannot be justified by any application of calculus of variations or optimization theory. Since optima are rather flat, performance losses for reasonable missions may be minor.

## IV. OPTIMIZATION CONSIDERATIONS

To compare the MSFC iterative guidance assumptions with MIT scheme assumptions, it is necessary to apply calculus of variations to the problem as formulated in the polar coordinate system by MIT. Applying calculus of variations to minimize fuel consumption to this set of equations results in the following set of equations which are to be solved for the optimum thrust direction $\alpha$:

$$\tan \alpha = \lambda_3/\lambda_4; \qquad \dot{\lambda}_3 = \lambda_1; \qquad \dot{\lambda}_4 = \lambda_2$$

$$-\dot{\lambda}_1 + \left\{\frac{r[\dot{V}_\theta + \dot{\lambda}_4 V_\theta] - V_\theta \lambda_4 \dot{r}}{r^2}\right\} = \lambda_3 \left[-\frac{2\mu}{r^3} + \frac{V_\theta^2}{r^2}\right] - \frac{\lambda_4 \dot{r} V_\theta}{r^2}$$

$$-\lambda_2 - \frac{2V_\theta}{r} + \frac{\dot{r}}{r}\lambda_4 = 0.$$

This set of equations is too complex to get a solution in closed form without making simplifying assumptions. The flat earth solution comes out easily by neglecting all terms involving division by r or $r^2$ giving the set

$$\dot{\lambda}_3 = \lambda_1; \quad \dot{\lambda}_4 = \lambda_2; \quad \dot{\lambda}_1 = 0, \quad \lambda_2 = 0$$

which have as a solution

$$\lambda_1 = a; \quad \dot{\lambda}_3 = a; \quad \lambda_3 = a + bt, \quad \lambda_4 = c.$$

Hence,

$$\tan \alpha = \frac{at + b}{c},$$

the familiar flat earth solution for unconstrained range and having two essential constants.

Some attempts will be made to justify the MIT approach from optimization considerations. If the flat earth solution is assumed to determine $\sin \alpha$ as time linear for the optimum thrust direction rather than the tangent as is true for small angles, then $\sin \alpha = a + bt$ may be assumed. This assumption renders the vertical equation of motion (1a) to the form

$$r = a_T(a + bt) - \frac{\mu}{r^2} + \frac{V_\theta^2}{r}.$$

A solution in closed form is still not possible because of the nonlinearity introduced by the gravitation and centrifugal terms. What MIT has done to get a closed solution may be considered in two ways.

1. Ignore or neglect the effective gravity term

$$-\frac{\mu}{r^2} + \frac{V_\theta^2}{r},$$

which anyway approaches zero as circular orbit is approached. Thus, the differential equation reduces to the form $\ddot{r} = a_T(a + bt)$, which may be integrated in closed form assuming constant thrust and mass flow. The coefficients a and b may be determined linearly from specified end conditions on r and $\dot{r}$ if the remaining time is known.

2. Define

$$\sin \alpha = -\left(\frac{\mu}{r^2} + \frac{V_\theta^2}{r}\right)/a_T + a + bt. \tag{24}$$

The result is the same as above. It will be noted that $\alpha$ is made up of two parts. The first part uses thrust to cancel or null the effective gravity term, an assumption leading to inefficient use of thrust for velocities much below orbital, the second having two degrees of freedom for the satisfaction of the two desired end conditions.

Although MIT describes their guidance principle as an optimizing guidance law, this can only be verified numerically in certain well chosen cases, as it seems to be (see Figure 2) for the Saturn IB trajectory example. A clear connection with optimization in theory has not been discovered unless the effects of significant forces are neglected. Rather, the assumptions of MIT are those of convenience in solving the boundary value problem. Since optima are rather flat, good performance can result in specific cases. MIT's basis for optimization is in selecting the proper forms for p(t) and q(t) in equation (8) where various functions of time have been considered. Calculus of variations does not determine p and q directly; it determines the control variable $\alpha$. The form assumed for equation (8) is not suggested by optimization theory without neglecting significant forces, but rather by the desire to make a linear boundary value problem which can be solved easily. Perhaps the multiple errors in applying the flat earth optimum solution to the MIT formulation are of a compensating nature.

The iterative scheme, on the other hand, operates in a coordinate system assumed inertial at any instant where equations of motion are simpler. The flat earth solution is simplified to $\tan \chi = \chi = a + bt$, thus, providing directly thrust angular rate as well as thrust angle. The iterative scheme fully accounts for the influence of gravitation in average over the flight the vehicle will subsequently experience. Thus, a closer connection with calculus of variations and optimization in some sensible degree of approximation may be established for the MSFC iterative guidance scheme than for the MIT principle. The MIT scheme seems to be based more on assumptions of convenience, although it shares many similarities with the MSFC scheme.

Figure 1 depicts the optimum thrust direction $\chi$ versus time on the Saturn IB calculus of variation trajectory computed over a spherical earth. The linearity of this function is obvious as is assumed by the iterative scheme. Also shown is $\alpha$ against the local vertical on the optimized trajectory. It is noted that no discontinuity is present at 161 seconds, LES jettison.

## V. MULTI-STAGE CONSIDERATIONS

Other possible differences between the iterative and the MIT scheme become apparent when generalization to multiple stages is contemplated. Studies in calculus of variations have shown that both $\chi$ and $\dot{\chi}$ are continuous at staging on optimized trajectories. The basic assumption of the iterative scheme, $\chi = a + bt$, easily adheres to this principle in multistage application. It is clear that, for the MIT scheme, $\alpha$ should be continuous at staging for optimization, where the better performing version, $\alpha$ is defined by equation (23). It is not easy to preserve the continuity of $\alpha$ since the coefficients of equation (22) are generally discontinuous at staging, as is $a_T$. It is to be expected that the MIT scheme is difficult to generalize to multiple stages in a way which preserves good performance optimization.* MIT guidance principles have been published for only a single stage. The iterative scheme, however, has been devised for and has been successfully demonstrated in multistage application where it has duplicated the results of calculus of variation computations to a close degree.

## VI. TIME-TO-GO ESTIMATION

Time to go until the proper end condition $V_{\theta D}$ is reached is determined by consideration of the differential equation (16), 16

$$\dot{V}_\theta = -\frac{\dot{r}V_\theta}{r} + a_T \cos \alpha .$$

* Intermediate end points may be used at the end of each stage destroying the adaptive nature of the scheme.

$$A = \frac{4}{T_{go}} [\dot{r}_D - \dot{r}_o] - \frac{6}{T_{go}^2} [r_D - (r_D + \dot{r}_o T_{go})] \tag{17a}$$

$$B = - \frac{6}{T_{go}} [\dot{r}_D - \dot{r}_o] - \frac{12}{T_{go}^3} [r_D - (r_D + \dot{r}_o T_{go})] \tag{17b}$$

and

$$\sin \alpha = \frac{1}{a_T} \left\{ A + B\ T_{go} - \left[ - \frac{\mu}{r^2} + \frac{V_\theta^2}{r} \right] \right\}. \tag{18}$$

For better performance, MIT chooses

$$p(t) = a_T = \frac{V_e}{\tau - t} \quad \text{with} \quad V_e = ge\ I_{sp};\ \tau = \frac{m_o}{\dot{m}} = \frac{V_e}{a_{To}} \tag{19a}$$

and

$$q(t) = a_T \cdot t. \tag{19b}$$

Thus, it is assumed that

$$\ddot{r} = Ca_T + Da_T \cdot t \tag{20}$$

characterizes optimum trajectories. The assumption is that $\ddot{r}/AT$ is time linear. This function for the Saturn IB optimized trajectory is shown in Figure 2. The linearity is quite good over the single stage except for the discontinuity at 161 seconds caused by LES jettison. Integrations called for in equations (9) and (10) may be carried out in closed form giving the steering equations.

A crude solution may be obtained by assuming the first term negligible since it goes to zero at circular orbit and assume $\alpha$ small so $\cos \alpha \approx 1$. Hence, an approximation may be obtained from

$$V_{\theta D} = V_{\theta o} = \int_{o}^{T_{go}} a_T \, dt; \quad a_T = \frac{V_e}{\tau - t}$$

$$V_{\theta D} - V_{\theta o} = - V_e \ln \left(1 - \frac{T_{go}}{\tau}\right) \tag{25}$$

which can be solved for $T_{go}$,

$$T_{go} = \tau \left\{1 - \exp \left[ - \left(\frac{V_{\theta D} - V_{\theta o}}{V_e}\right)\right]\right\}. \tag{26}$$

This equation gives an approximation of $T_{go}$ with an initial error of about 2 percent which goes to zero as the end condition is approached on the Saturn IB trajectory selected for examination. It seems this simple approach is worthy of investigation.

MIT has presented a different approach requiring an undesirable iteration in flight.

Equation (16) is rewritten as

$$\dot{V}_{\theta} = a_T + \left[a_T (\cos \alpha - 1) - \frac{\dot{r} V_o}{r}\right].$$

Integrating both sides between 0 and t gives

$$V_{\theta}(t) - V_{\theta}(o) = \int_{o}^{t} \frac{V_e}{\tau - t} \, dt + \Delta V_L \tag{27a}$$

$$V_\theta(t) - V_\theta(o) = - V_e \ln (1 - \frac{t}{\tau}) + \Delta V_L, \tag{27b}$$

where $\Delta V_L$ represents the integral of the acceleration terms in brackets. If the minor $\Delta V_L$ term is neglected, then

$$\exp - \left[\frac{V_\theta(t) - V_\theta(o)}{V_e}\right] = 1 - \frac{t}{\tau} \tag{28}$$

is a simple linear approximation to the relationship between the velocities at any later time t. If equation (27) is evaluated at $T_{go}$, when desired end conditions are satisfied and solved for $T_{go}$, then

$$T_{go} = \tau \left\{1 - \exp \left[- \frac{(V_{\theta D} - V_{\theta o} + \Delta V_{\theta L})}{V_e}\right]\right\} \tag{29}$$

results. Everything in equation (29) is known from engine characteristics except $\Delta V_L$ and $T_{go}$. It is necessary to predict or to guess at $\Delta V_L$. For this purpose a function analogous to equation (28) is defined for expansion in Taylor's series.

$$H(t) = \exp \left\{- \frac{[V_\theta(t) - V_{\theta o}]}{V_e}\right\}. \tag{30}$$

The simplified equation (28) shows that this function principally is linear in time and therefore should be capable of a simple truncated Taylor expansion.

$$H(t) = H_o + \dot{H}_o(t - t_o) + \frac{\ddot{H}}{2} (t - t_o)^2 +, \dots . \tag{31}$$

The function H is used to predict $V_\theta$ at $T_{go}$ after a guess has been made for the value $\Delta V_L$, and $T_{go}$ is computed by equation (29). If

$$V_{\theta D} = V_\theta(T_{go})$$

is not the desired value, then $\Delta V_L$ must be incremented and a new guess, and a new $T_{go}$ obtained. The process is repeated until a satisfactory degree of convergence on $V_{\theta D}$ is obtained. Once convergence is obtained and $T_{go}$ computed, the steering equations for that computation cycle may be evaluated. Objections to this process may be raised on two grounds. One is a general objection to iterative processes in flight. Another is that the expansion (31) requires $\dot{\alpha}$ in H and higher derivatives of angle of attack or of $\dot{V}_\theta$ which are complicated and perhaps difficult to obtain in flight.

The recourse for a non-iterative $T_{go}$ determination is to use the simplified equation (26) alone or with a simple single-step correction process. The defect of equation (26) is that it does not consider the $\Delta V_L$ term, but assumes $\Delta V_L = 0$. Define the bracketed acceleration term

$$\Delta\dot{V}_L = a_T\ (\cos\ \alpha - 1) - \frac{\dot{r}V_\theta}{r} \approx \frac{a_T\alpha^2}{2} - \frac{\dot{r}V_\theta}{r} \tag{32}$$

which is quite nonlinear over the trajectory. A crude numerical integration of $\Delta V_L$ using the value of $\Delta\dot{V}_L$ at two points, the current time and the desired end condition, can be attempted where the end conditions are obtained from the desired orbit. If injection is into circular orbit, $\dot{r}$ on the orbit is zero and $\alpha$ terminal is zero; hence, $\Delta\dot{V}_{LD} = 0$.* Time to go obtained from equation (26), and the associated $\alpha$ will be called preliminary values. The following outline defines a single step improvement process:

1. Compute $T_{go}P$ from equation (26)

2. Compute $\alpha P$ from steering equations

3. Compute $\Delta V_L$ at the current time

4. Compute $\Delta V_L = 1/2\ T_{go}P\ [\Delta\dot{V}_{Lo} + \Delta\dot{V}_{LD}]$ with $\Delta\dot{V}_{LD} = 0$

5. Convert $\Delta V_L$ to burning time with the ideal velocity equation. Since $\Delta V_L$ is small compared with $V_e$, the approximation

$$\Delta T_{go} = \frac{\tau\ \Delta V_L}{V_e} \tag{33}$$

can be used.

* This assumption is unrealistic, however, it furnishes a partial correction of $T_{go}$.

$$T_{go} = T_{go}p + \Delta T_{go} \tag{34}$$

6. The change in $\alpha$ corresponding to $\Delta T_{go}$:

$$\Delta \sin \alpha = D\ \Delta T_{go}. \tag{35}$$

$$\sin \alpha = \sin \alpha p + \Delta \sin \alpha. \tag{36}$$

Thus, improved numerical values can be obtained in a single-step process without involving iteration in flight. For injection into moderately elliptical orbits, $\dot{r}$ at injection on the orbit is small and the process may still suffice. For injection into highly eccentric orbits, a reasonable value for $\Delta\dot{V}_{LD}$ may be preset from a nominal trajectory. Since only a correction process is being described, a high accuracy is not required; also, any errors in $T_{go}$ prediction go to zero as the remaining time reduces. As an alternate process, one may replace step 4 with

$$\Delta V_L = k\ T_{go}p\ \Delta\dot{V}_{Lo},$$

where numerical studies can be used to determine a good value for k which may be held constant over flight time.

MIT has made a strong point that the time-to-go iteration process they have proposed does not require the evaluation of the exponential function. Some RAND polynomials are given in Appendix A which shows that the evaluation of transcendental functions can be accomplished with good accuracy using a minimum of digital computer storage without requiring lengthy series expansions. Since the guidance scheme is a null seeking device, a high accuracy is not required.

## VII. GUIDANCE EQUATIONS ARRANGED FOR COMPUTATION

It is assumed that the IMU system provides true position and velocity x, y, x, y in a rectangular coordinate system oriented in the usual manner relative to the launch site. Values at the current state of flight are indicated by subscript 0 and by subscript D at the desired terminal state. It is also assumed that IMU and associated digital computer system provide the total engine acceleration $a_t$. Equations for the pitch plane guidance of a single stage in vacuum follow.

It is necessary to transform rectangular coordinate information to the polar system used by MIT. The equations of transformation are

$$r^2 = (re + y)^2 + x^2; \quad re = \text{earth radius}$$

$$\sin \varphi = \frac{x}{r}; \quad \cos \varphi = \frac{re + y}{r}$$

$$\dot{r} = \dot{x} \sin \varphi + \dot{y} \cos \varphi$$

$$V_\theta = \dot{x} \cos \varphi - \dot{y} \sin \varphi$$

$$\varphi = \tan^{-1} \frac{x}{re + y}.$$

The velocity

$$V^2 = \dot{x}^2 + \dot{y}^2$$

may be needed in the vicinity of cutoff if cutoff is based on velocity; otherwise, cutoff can be computed from $T_{go} \to 0$.

Once a thrust direction $\alpha$ is obtained relative to the local horizontal, it is necessary to transform to the platform horizontal reference

$$\chi = \alpha - \varphi.$$

<u>Simplified Time-To-Go</u>

The inputs are $V_{\theta D}$, $V_{\theta o}$, $a_T$.

$$T_{go} = \tau \left\{1 - \exp\left[-\left(\frac{V_{\theta D} - V_{\theta o}}{V_e}\right)\right]\right\}; \ \tau = \frac{V_e}{a_T}; \ V_e = ge\ I_{sp},$$

with $V_e$ nominal.

<u>Steering command equations follow for the better performing scheme:</u>

The inputs are $r_D$, $r_o$, $\dot{r}_D$, $\dot{r}_o$, $T_{go}$, $a_T$.

$$a_{11} = - V_e \ln\left(1 - \frac{T_{go}}{\tau}\right)$$

$$a_{12} = a_{12}\tau - V_e\ T_{go}$$

$$a_{21} = - a_{12} + a_{11}\ T_{go}$$

$$a_{22} = a_{21}\tau - \frac{1}{2} V_e\ T_{go}^2$$

$$\Delta = a_{11}\ a_{22} - a_{12}\ a_{21}$$

$$b_{11} = \frac{a_{22}}{\Delta};\ b_{12} = - \frac{a_{12}}{\Delta};\ b_{21} = - \frac{a_{21}}{\Delta};\ b_{22} = \frac{a_{11}}{\Delta}$$

$$C = b_{11}\ [\dot{r}_D - \dot{r}_o] + b_{12}\ [r_D - r_o - \dot{r}_o\ T_{go}]$$

$$D = b_{21}\ [\dot{r}_D - \dot{r}_o] + b_{22}\ [r_D - r_o - \dot{r}_o\ T_{go}]$$

$$\sin \alpha = - \frac{1}{a_T} \left[ - \frac{\mu^2}{r^2} + \frac{V_\theta^2}{r} \right] + C + Dt. \tag{37}$$

$$\alpha = \sin^{-1} (\sin \alpha).$$

Stop computation of $\alpha$ and hold constant when $T_{go}$ becomes small.

The computation is arranged in a major computing cycle and a minor cycle. A major cycle involves the entire set of equations and determination of C and D. The minor cycle updates $\alpha$ from equation (37) using previously determined C and D. In equation (37), t is the time elapsed since the last determination of C and D.

## VIII. CONCLUSIONS

At this point, some conclusions may be drawn in comparing the MIT guidance principle with the MSFC iterative guidance.

a. The MIT scheme uses thrust to cancel out the effective gravity term enabling integration of the $\ddot{r}$ equation. This is an inefficient use of thrust for velocities much different from orbital. The iterative guidance scheme, on the other hand, does not neglect or compensate significant forces with thrust for the sake of linearity; rather, significant forces are taken into account by the scheme, and linearity is obtained by other means.

b. The iterative guidance scheme is a natural outgrowth of the calculus of variation solution over a flat earth adjusted to take into account the variation of gravity over a spherical earth. The MIT principle cannot be connected with optimization theory in any sensible degree of approximation analytically. Since optima are not sharp, it is likely that the MIT principle can deviate from optimum by insignificant amounts in some specific cases.

c. It is not clear how to generalize the MIT principle to multistage application preserving the continuity considerations which must hold at staging on optimized trajectories. The iterative guidance scheme is based on principles derived from calculus of variations which are readily extended to multiple stages as has been demonstrated in the past. Intermediate and points at the end of each stage may be used for the MIT scheme; however, these tend to destroy the adaptive nature of the scheme and are difficult to select so that strong control discontinuities are not introduced at staging. The MIT scheme deserves consideration in single stage applications where computation in flight must be minimized, and for specific missions where its performance characteristics are satisfactory. However, MSFC iterative guidance computer requirements are also moderate.

d. The functions to be performed by MIT guidance equipment and schemes include service module braking into lunar orbit, LEM descent and ascent, and possible launch vehicle backup in the event of primary guidance failure. The MSFC iterative guidance scheme has demonstrated its ability in all these applications, even those applications over multiple stages.

e. The MIT scheme provides only thrust direction and not angular rate also, as does the iterative guidance scheme. This either requires additional computation to provide angular rate for the updating of $\alpha$ in between major computing cycles or occurrence of the minor computing cycles at small intervals requiring the arc sine evaluation each time. The time-to-go estimation as proposed by MIT requires an undesirable iteration in flight, unless the simplification discussed in this report or something similar is adopted.

f. Stability of any guidance scheme is an important consideration and is primarily a function of thrust over weight ratio. Stability has not been carefully investigated at this time, however, preliminary investigations point to a possible instability problem with the MIT scheme to a greater extent than is present for the iterative scheme.

Page 20

blank page (left hand side)

# APPENDIX A

## RAND POLYNOMIALS FOR THE GENERATION OF TRANSCENDENTAL FUNCTIONS

Reference: Approximations for Digital Computers, Hastings, Princeton University Press, 1955.

Polynomial curve fits for several of the transcendental functions involved together with the range of independent variable and maximum error taken from the reference are given. The error is oscillatory in nature and is zero at several points over the range, the number of zeroes depending on the degree of the approximating expression.

$$\sin^{-1} x = \frac{\pi}{2} - \sqrt{1 - x} \cdot \chi(x) \qquad 0 \leqq x \leqq 1$$

$$\chi(x) = 1.5707228 - .2121144\, x + .0742610x^2 - .0187293x^3$$

error $< 7 \times 10^{-5}$ radians $\approx .004°$

$$e^{-x} = \frac{1}{[1 + .2507213x + .0292732x^2 + .0038278x^3]^4} \qquad 0 \leqq x \leqq \infty$$

error $< 3 \times 10^{-4}$

$$\ln(1 + x) = .9974442 - .4712839x^2 + .2256685x^3 - .0587527x^4$$

$0 \leqq x \leqq 1$ error $< 7 \times 10^{-5}$.

Formulas of greater accuracy can be found in the reference.

Page 22
Lft hand (blank) page

# APPENDIX B

## PERFORMANCE COMPARISON OF MIT SCHEME WITH CALCULUS OF VARIATIONS

The Saturn IB trajectory selected for comparison has been run starting just after LES jettison time at 161 seconds using the MIT guidance principle with simplified time-to-go prediction. Beginning and end conditions are the same as those used on a comparable calculus of variations run. The calculus of variation trajectory run exhibited 5 pounds greater payload into the 105-mile orbit. Position and velocity errors at cutoff on the MIT run were negligible. This would be true of any reasonably perturbed trajectories since the MIT scheme is closed around the end point. Figure 4 shows the deviation of the MIT angle X against that obtained with the calculus of variations trajectory run. The comparison has been made using the same type of numerical integration and integration step size in both cases. This example seems to indicate that the multiple errors of assumption in the MIT approach are compensatory in nature.

MIT runs have been made on the ASI computer, Mr. Hollis Arban programmer.

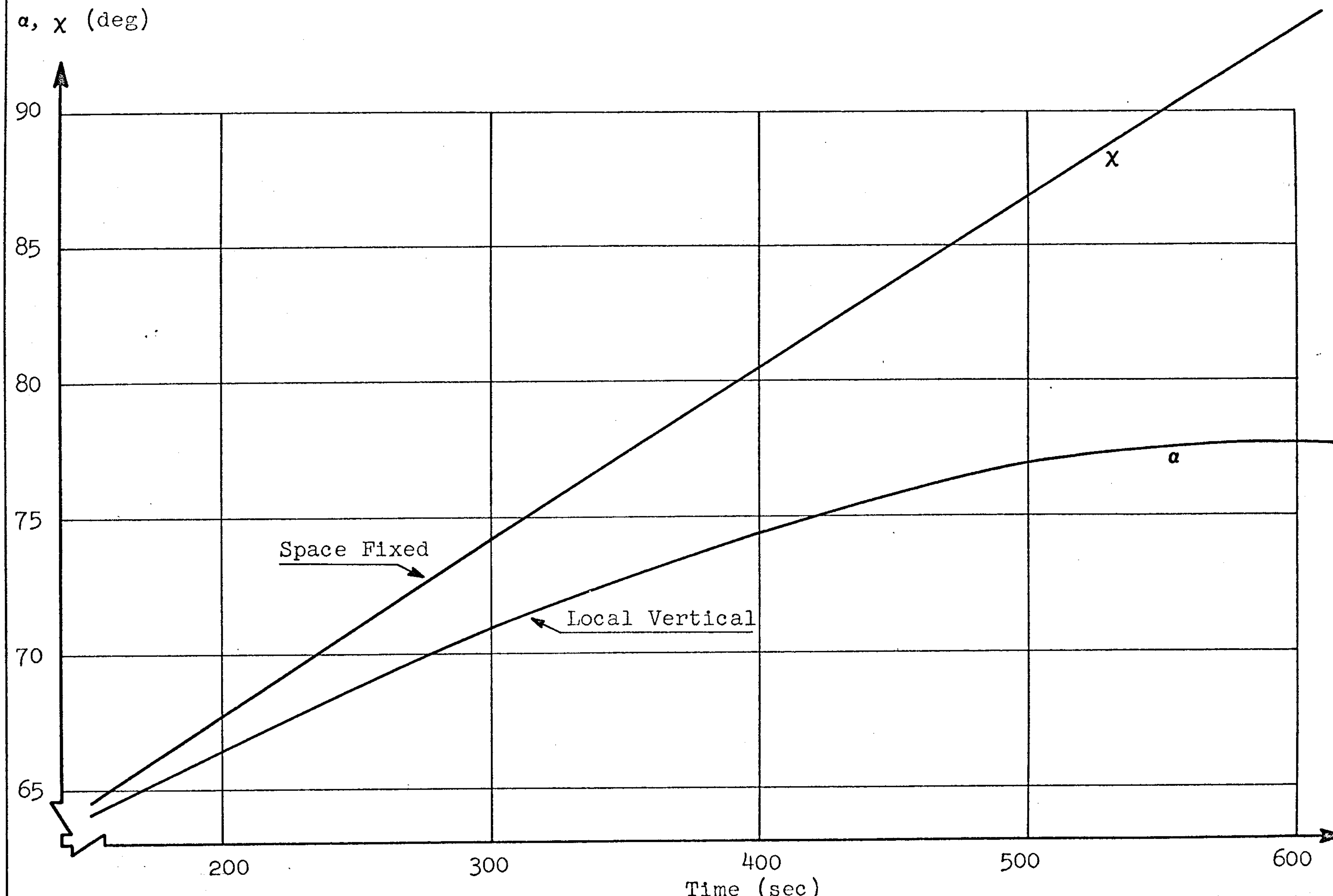


FIG. 1. THRUST DIRECTION VERSUS TIME

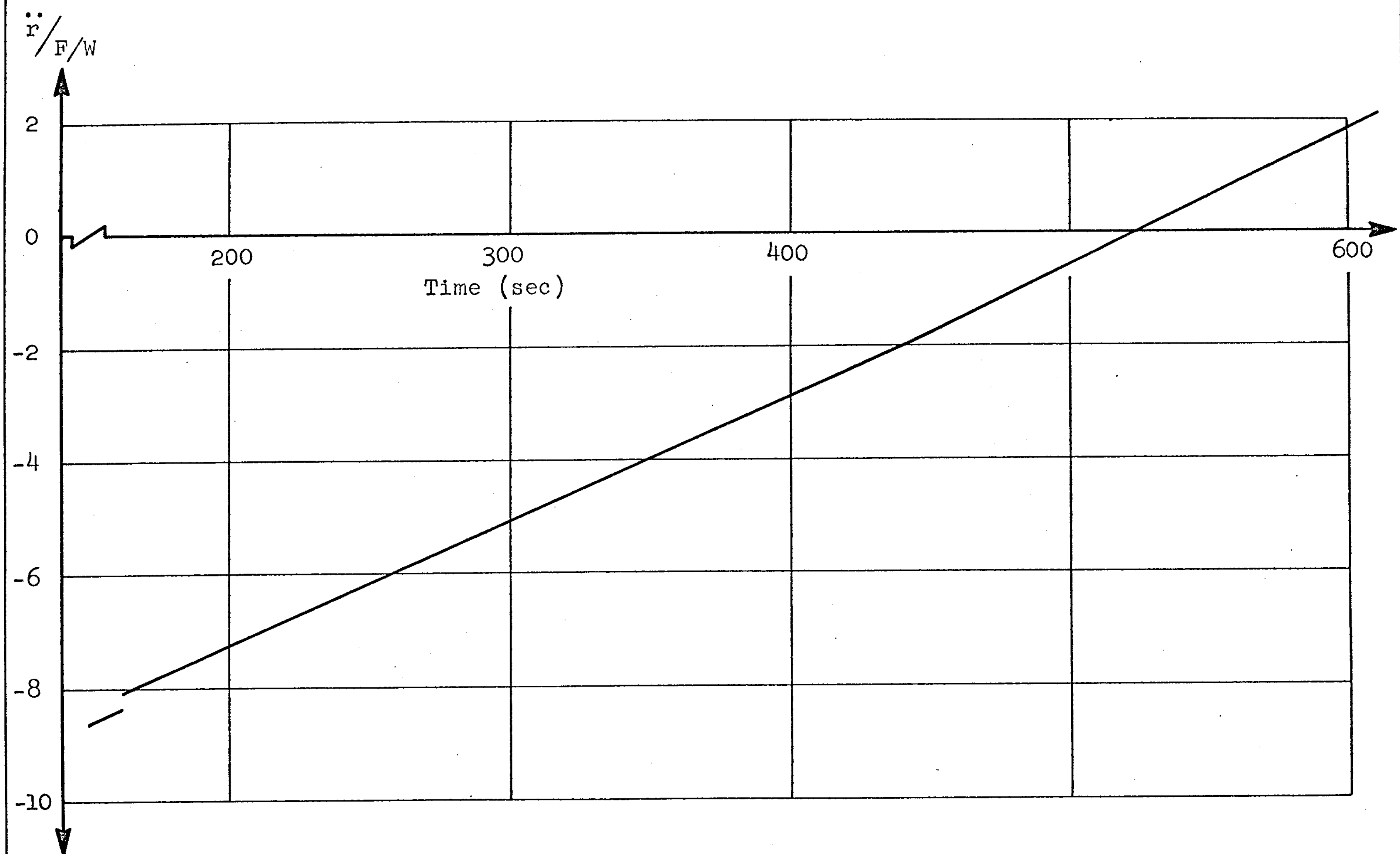


FIG. 2. $\ddot{r}$ DIVIDED BY THRUST OVER WEIGHT RATIO VERSUS TIME

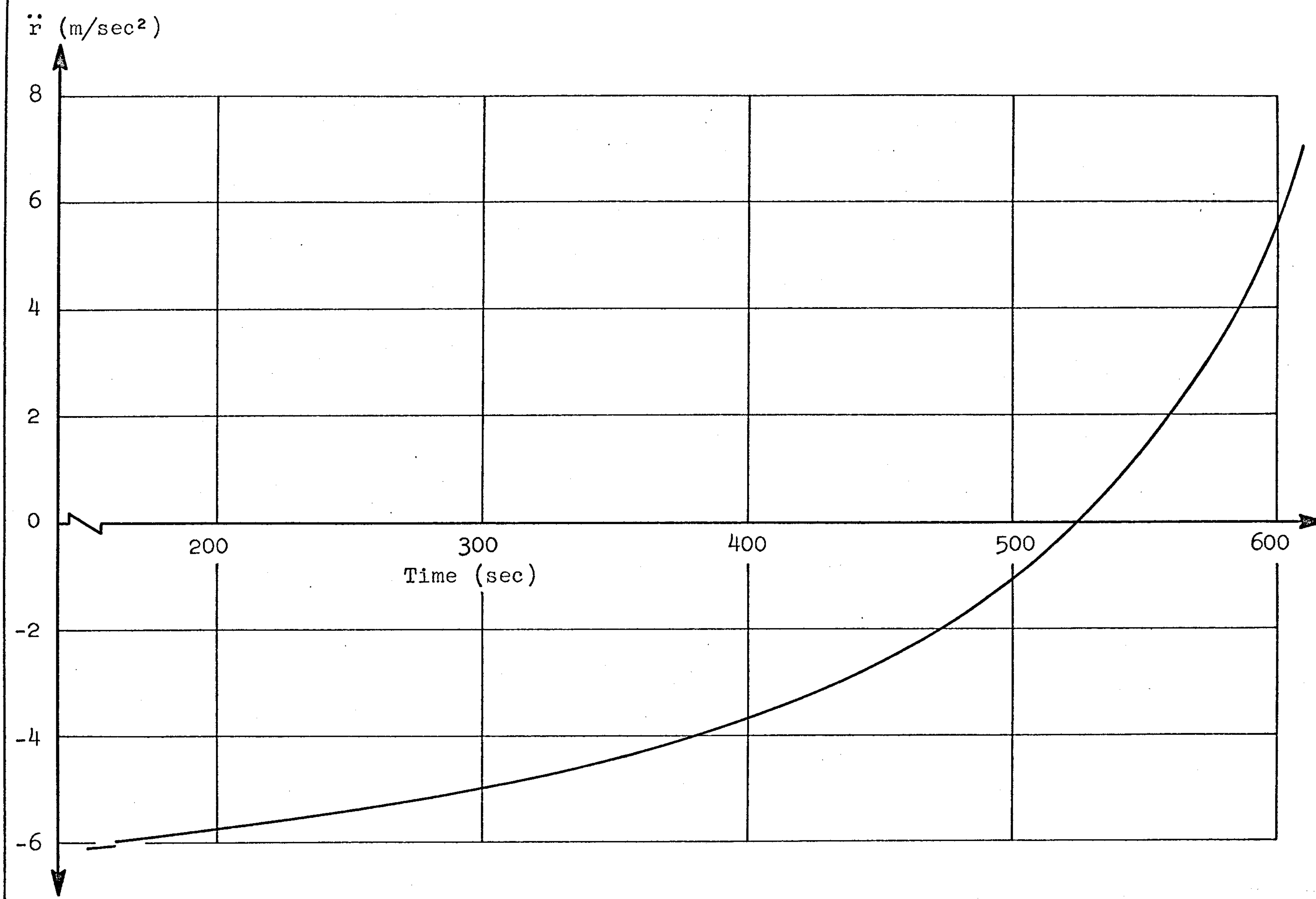


FIG. 3. $\ddot{r}$ VERSUS TIME

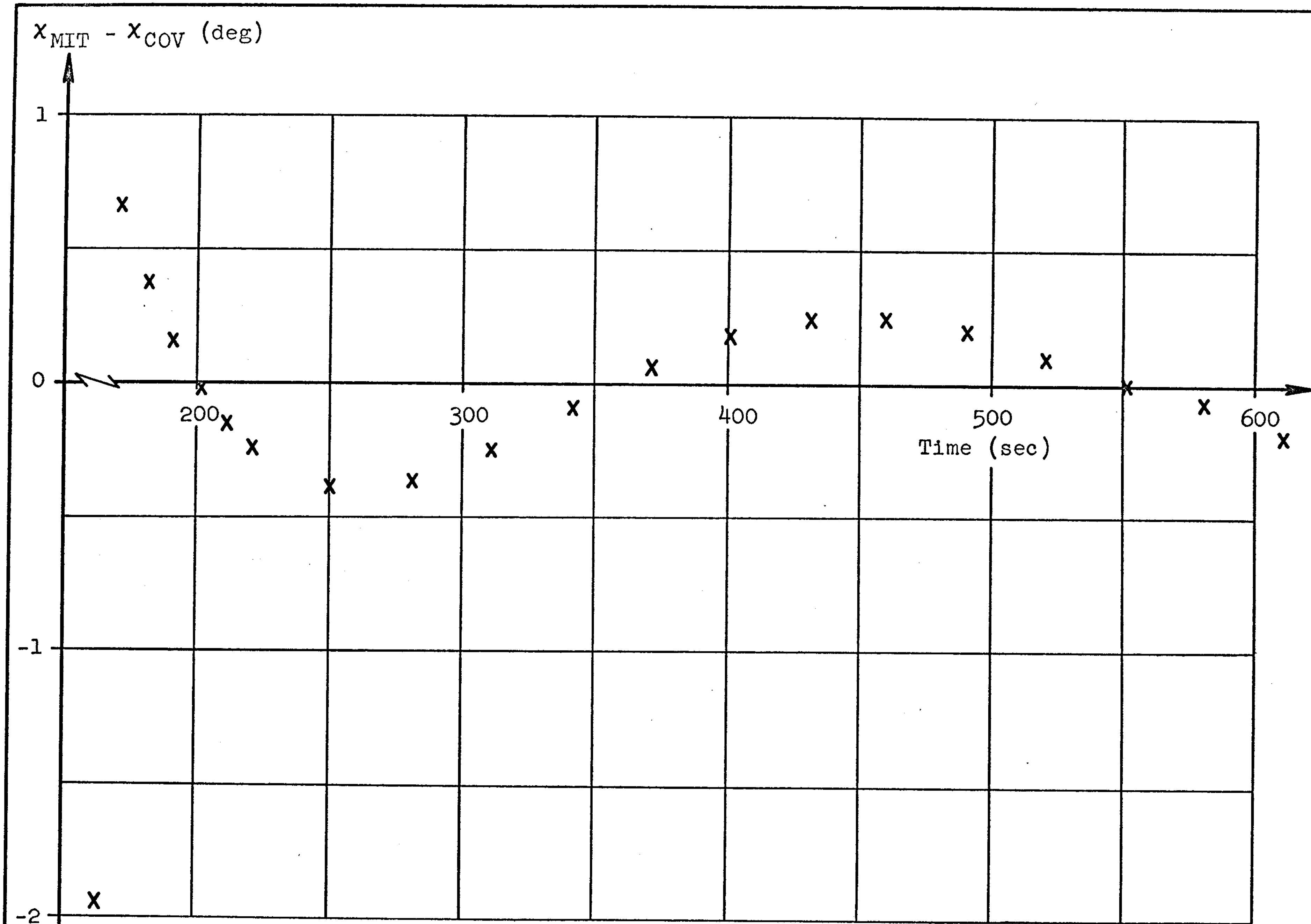


FIG. 4

## DISTRIBUTION

R-AERO
- Dr. Geissler
- Mr. Horn
- Mr. Baker
- Mr. Hart
- Mrs. Chandler
- Mr. Winch
- Mr. Telfer
- Mr. Arban
- Mr. Teuber
- Mr. Rheinfurth

R-ASTR
- Dr. Haeussermann
- Mr. Brooks Moore
- Mr. Seltzer
- Mr. Melvyn Brooks

APPROVAL R-AERO-IN-23-64

A COMPARISON OF AN MIT EXPLICIT GUIDANCE PRINCIPLE
WITH MSFC ITERATIVE GUIDANCE

By

Judson J. Hart

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

Helmut J. Horn
Chief, Dynamics and Flight Mechanics Division

E. D. Geissler
Director, Aero-Astrodynamics Laboratory