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DECISION PROCEDURE FOR MINIMIZING COSTS OF CALIBRATING LIQUID ROCKET ENGINES

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SUMMARY:

Prior to acceptance of a liquid rocket engine for use in Saturn vehicles, the average thrust of two consecutive tests without an intervening calibration must satisfy specification requirements. The contractor may recalibrate after the first and subsequent tests if he so chooses, based upon decision limits, until the above requirement is met.

This paper provides a method for calculating decision limits such that the total number of tests required for acceptance is minimized. The model for calculating the decision limit takes into account operational reliability and life of the engine, ratio of cost of testing to cost of an engine, and correlation between tests as a function of engine-to-engine and run-to-run variance components.

INTRODUCTION:

One of the requirements for NASA acceptance of a Saturn vehicle engine is that the thrust averaged from two successive tests without an intervening calibration fall within specification limits. In the past, most engines were accepted from the contractor after three tests, but when the specification was recently tightened it was estimated that more than 50% of all engines would have to be tested at least four times prior to acceptance. Their increase in number of tests per engine represented an appreciable increase in costs.

This paper presents the results of a study made to determine what could e done to reduce acceptance testing costs when the specification limits are held constant.

DISCUSSION:

Engine testing is conducted in accordance with the following ground rules until the engine meets acceptance requirements or until it is scrapped:

1. If thrust in a test following a calibration is outside certain decision limits, the engine is successively recalibrated and tested until thrust falls inside the decision limits.
2. If thrust in a test following a calibration is inside the decision limits, no changes are made to the engine and another test is conducted in an attempt to satisfy acceptance requirements.

3. If the average thrust from two consecutive tests without intervening calibration falls outside of specification limits, the engine is recalibrated, and the test cycle is repeated.

It should be pointed out that the value in using a procedure such as described below is greatest when specification limits are tight. If specification limits are very wide, there is not much point in using decision limits at all, because the need for recalibration becomes remote.

ILLUSTRATIONS:

For the purpose of applications herein, the following assumptions were made:

1. The engine is always calibrated after the first test (due to high variability of thrust prior to the first calibration).
2. There is no bias introduced in calibrating the engine.
3. After the initial calibration, ability to recalibrate does not improve between tests.
4. Cost of calibration is negligible compared to cost of a test.
5. The engine is scrapped after N tests that do not satisfy the criterion for acceptance described above.
6. The engine-to-engine and run-to-run variance components, σ^2_{EE} and σ^2_{RR}, respectively, are known; the mean thrust is also known.

7. σ^2_{RR} is the same for all engines.

8. Engine-to-engine and run-to-run deviates are normally and independently distributed.

The models described below can easily be altered to change assumptions 1 through 5.

Two models are considered herein:

1. Assume the engine is scrapped after nine unsuccessful tests, and operational reliability = 1.0. Operational reliability is defined as one minus the probability of any failure (hardware, facility, human error) that causes a single additional test and calibration. Assume that the cost of scrapping an engine is equal to the cost of 40 tests.
2. Assume the engine is scrapped after 5 unsuccessful tests and operational reliability ≤ 1.0.

Common to all models generated under the above assumptions, we define the following probabilities (figure 1):

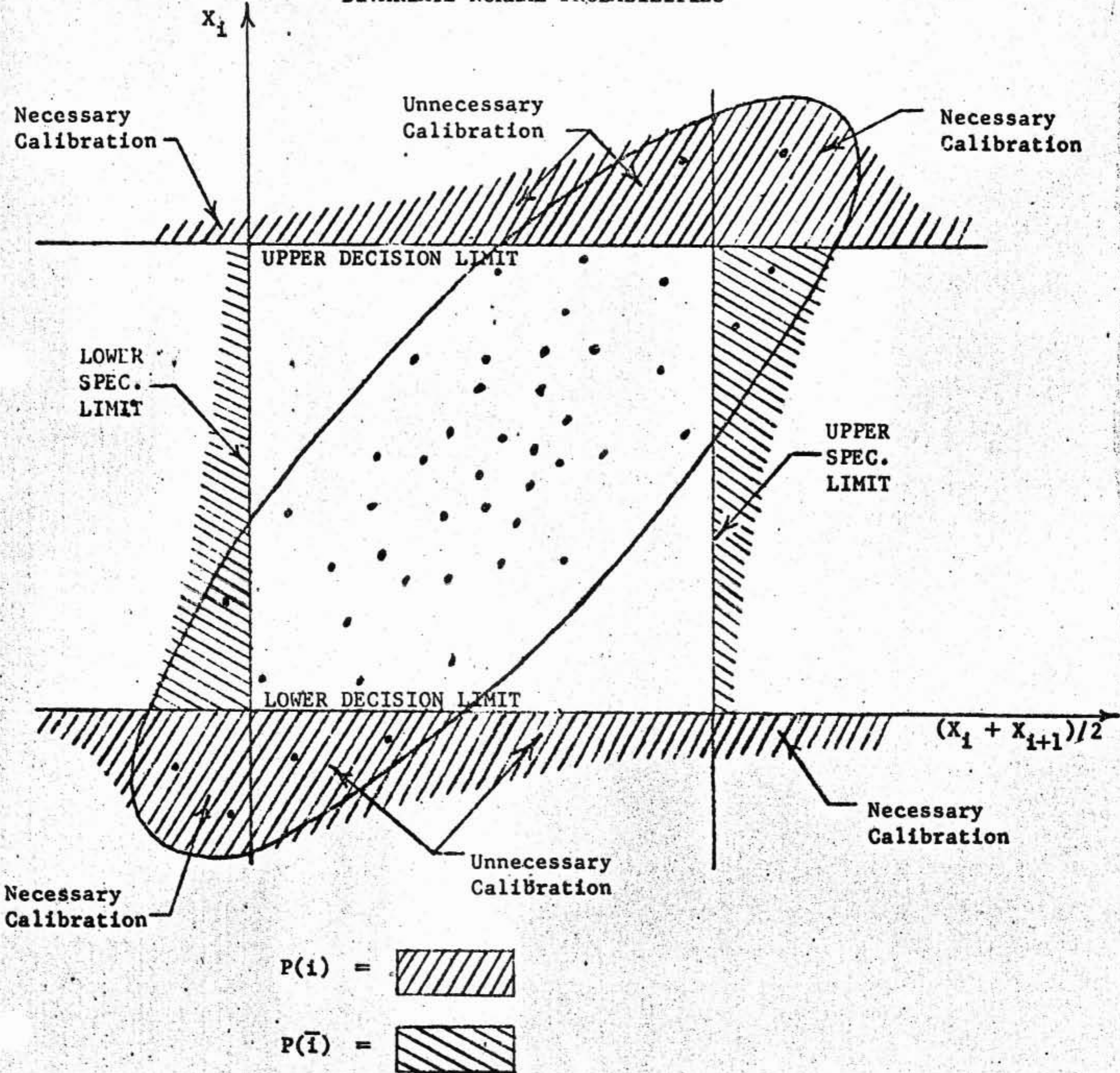
Let $P(i)$ be the probability of thrust exceeding the decision limits in the i^{th} test.

Let $P(\bar{i})$ be the conditional probability that the mean thrust, $(X_i+X_{i+1})/2$, of the i^{th} and $(i+1)^{th}$ tests exceeds the specification limits.

It is assumed that $P(i)$ is the same for all i, and that $P(\bar{i})$ is the same for all i. Assuming normality, P(i) and P($\overline{i}$) may be calculated from the bivariate normal density as illustrated in figure 1.

Figure 1

BIVARIATE NORMAL PROBABILITIES



$P(i)$ and $P(\bar{i})$ may be obtained from equations (1), (2), (3) below by using any table of the bivariate normal distribution, such as reference (1). It is convenient to express the correlation coefficient as a function of the run-to-run and engine-to-engine variance components, because of the advantage gained by utilizing all pertinent data. From the appendix, the standard deviation of X_i is :

$$\sigma_{X_i} = \sqrt{\sigma_{RR}^{2} + \sigma_{EE}^{2}} \qquad (1)$$

The correlation coefficient between X_i and $(X_i+X_{i+1})/2$ is:

$$\rho_{X_i,\ (X_i+X_{i+1})/2} = \sqrt{1 - \frac{\sigma_{RR}^{2}}{2\,\sigma_{X_i}^{2}}} \qquad (2)$$

The standard deviation of $(X_i+X_{i+1})/2$ is:

$$\sigma_{(X_i+X_{i+1})/2} = \sigma_{X_i}\,\rho_{X_i,\ (X_i+X_{i+1})/2} \qquad (3)$$

MODEL 1: Reliability = 1.0; engine is scrapped after 9 unsuccessful tests.

Let the notation "$2\ \bar{3}$" describe the event that thrust of the second test was within decision limits and that the average thrust of the second and third test was within specification limits. Let the notation "$2_c\ 3\ \bar{4}_c\ 5\ \bar{6}$" describe the following event:

- Calibration after second test (thrust outside of decision limits).
- No calibration after third test (thrust within decision limits).
- Calibration after fourth test (mean thrust of third and fourth test outside of specification limits).
- Thrust in fifth test within decision limits.
- Average thrust of fifth and sixth tests within specification limits.

Using this notation and the notation of figure 1, probabilities for the various events are as follows:

TABLE I

EVENTS	PROBABILITY
2 $\bar{3}$	$1-P(1)-P(\bar{1})$
2_c 3 $\bar{4}$	$[1-P(1)-P(\bar{1})]\,P(1)$
2 $\bar{3}_c$ 4 $\bar{5}$	$[1-P(1)-P(\bar{1})]\,P(\bar{1})\,[1-P(1)]$
2_c 3_c 4 $\bar{5}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^2$
2 $\bar{3}_c$ 4_c 5 $\bar{6}$	$[1-P(1)-P(\bar{1})]\;P(1)\;P(\bar{1})\;[1-P(1)]$
2_c 3 $\bar{4}_c$ 5 $\bar{6}$	$[1-P(1)-P(\bar{1})]\;P(1)\;P(\bar{1})\;[1-P(1)]$
2_c 3_c 4_c 5 $\bar{6}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^3$
2 $\bar{3}_c$ 4 $\bar{5}_c$ 6 $\bar{7}$	$[1-P(1)-P(\bar{1})]\,[P(\bar{1})]^2\,[1-P(1)]^2$
2 $\bar{3}_c$ 4_c 5_c 6 $\bar{7}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^2\,P(\bar{1})[1-P(1)]$
2_c 3 $\bar{4}_c$ 5_c 6 $\bar{7}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^2\,P(\bar{1})[1-P(1)]$
2_c 3_c 4 $\bar{5}_c$ 6 $\bar{7}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^2\,P(\bar{1})[1-P(1)]$
2_c 3_c 4_c 5_c 6 $\bar{7}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^4$
2 $\bar{3}_c$ 4 $\bar{5}_c$ 6_c 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\;P(1)\;[P(\bar{1})]^2[1-P(1)]^2$
2 $\bar{3}_c$ 4_c 5 $\bar{6}_c$ 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\;P(1)\;[P(\bar{1})]^2[1-P(1)]^2$
2 $\bar{3}_c$ 4_c 5_c 6_c 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^3\;P(\bar{1})\;[1-P(1)]$
2_c 3 $\bar{4}_c$ 5 $\bar{6}_c$ 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\;P(1)\;[P(\bar{1})]^2\,[1-P(1)]^2$
2_c 3 $\bar{4}_c$ 5_c 6_c 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^3\;P(\bar{1})\;[1-P(1)]$
2_c 3_c 4 $\bar{5}_c$ 6_c 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^3\;P(\bar{1})\;[1-P(1)]$
2_c 3_c 4_c 5 $\bar{6}_c$ 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^3\;P(\bar{1})\;[1-P(1)]$
2_c 3_c 4_c 5_c 6_c 7 $\bar{8}$	$[1-P(1)-P(\bar{1})]\,[P(1)]^5$

TABLE I (Cont'd)

EVENTS	PROBABILITY
$2\ \bar{3}_c\ 4\ \bar{5}_c\ 6\ \bar{7}_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(\bar{i})]^3\,[1 - P(i)]^3$
$2\ \bar{3}_c\ 4\ \bar{5}_c\ 6_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^2\,[P(\bar{i})]^2\,[1 - P(i)]^2$
$2\ \bar{3}_c\ 4_c\ 5\ \bar{6}_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^2\,[P(\bar{i})]^2\,[1 - P(i)]^2$
$2\ \bar{3}_c\ 4_c\ 5_c\ 6\ \bar{7}_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^2\,[P(\bar{i})]^2\,[1 - P(i)]^2$
$2\ \bar{3}_c\ 4_c\ 5_c\ 6_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^4\,P(\bar{i})\,[1 - P(i)]$
$2_c\ 3\ \bar{4}_c\ 5\ \bar{6}_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^2\,[P(\bar{i})]^2\,[1 - P(i)]^2$
$2_c\ 3\ \bar{4}_c\ 5_c\ 6\ \bar{7}_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^2\,[P(\bar{i})]^2\,[1 - P(i)]^2$
$2_c\ 3\ \bar{4}_c\ 5_c\ 6_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^4\,P(\bar{i})\,[1 - P(i)]$
$2_c\ 3_c\ 4\ \bar{5}_c\ 6\ \bar{7}_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^2\,[P(\bar{i})]^2\,[1 - P(i)]^2$
$2_c\ 3_c\ 4\ \bar{5}_c\ 6_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^4\,P(\bar{i})\,[1 - P(i)]$
$2_c\ 3_c\ 4_c\ 5\ \bar{6}_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^4\,P(\bar{i})\,[1 - P(i)]$
$2_c\ 3_c\ 4_c\ 5_c\ 6\ \bar{7}_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^4\,P(\bar{i})\,[1 - P(i)]$
$2_c\ 3_c\ 4_c\ 5_c\ 6_c\ 7_c\ 8\ \bar{9}$	$[1 - P(i) - P(\bar{i})]\,[P(i)]^6$

Assume that the cost of one engine is equivalent to the cost of $M = 40$ tests. Let P_j be the sum of probabilities in table 1 associated with those events requiring j tests. Then the expected number of tests per accepted engine is:

$$E(N) = \frac{\sum_{j=3}^{N} jP_j + M(1 - \sum_{j=3}^{N} P_j)}{\sum_{j=3}^{N} P_j} \qquad (4)$$

The quantity in parenthesis is the probability that more than 9 tests are required; i.e. , the probability of scrapping the engine. Holding the specification limit constant, the decision limit (figure 1) is varied until E(N) is minimized.

In illustration, this model was used to support contract negotiations in an engine program where reliability of the engine is very high. Practice is to scrap the engine after 9 unsuccessful tests. Data showed that the square root of the within-engine, or run-to-run variance component of thrust was 600 lbs., and the square root of the engine-to-engine variance component was between 1200 and 1500 lbs. Both extremes were analyzed, as follows:

<u>Case 1</u>: $\sigma_{EE} = 1200$ lbs. $\sigma_{RR} = 600$ lbs.

From equation (1), $\sigma_{X_i} = \sqrt{(600)^2 + (1200)^2} = 1340$ lbs

From equation (2) $\rho_{X_i, (X_i + X_{i+1})/2} = \sqrt{1 - \frac{1}{2}\left(\frac{600}{1340}\right)^2} = .95$

From equation (3) $\sigma_{(X_i + X_{i+1})/2} = .95(1340) = 1270$ lbs

Suppose the specification limits for thrust are nominal $\pm$ 2000 lbs. Then the number of standard deviations between nominal and the specification limit (two-sided) is 2000/1270 = 1.57. By trial and error, equation (4) is minimized when the decision limits are nominal $\pm$ 1.7(1340) = nominal $\pm$ 2200 lbs, when E(N) = [3.178 + 40(.0020)] /.998 = 3.26 tests per accepted engine.

<u>Case 2</u>: $\sigma_{EE} = 1500$ lbs $\sigma_{RR} = 600$ lbs

From equation (1), $\sigma_{X_i} = \sqrt{(600)^2 + (1500)^2} = 1620$ lbs

From equation (2), $\rho_{X_i, (X_i + X_{i+1})/2} = \sqrt{1 - \frac{1}{2}\left(\frac{600}{1620}\right)^2} = .965$

From equation (3), $\sigma_{(X_i + X_{i+1})/2} = .965(1620) = 1560$ lbs

The number of standard deviations between nominal and the specification limit (two-sided) is 2000/1560 = 1.28. By trial and error, equation (4) is minimized when the decision limits are nominal $\pm$ 1.5(1620) = nominal $\pm$ 2430 lbs, when E(N) = [3.286 + 40(.0122)] /.988 = 3.8 tests per accepted engine. (Note that changing the ratio of σ_{RR}/σ_{X_i} from 600/1340 in case 1 to 600/1620 in case 2 changes the correlation coefficient by only .015, and merely changes the optimum decision limits from 1.7 to 1.5 standard deviations. E(N) changes significantly, from 3.3 to 3.8 tests per accepted engine.)

Other information of interest corresponding to decision limits is the following:

A. Prob. of acceptance after N tests = $\sum_{j=3}^{N} P_j$

B. Prob. of scrapping engine after N tests = $1 - \sum_{j=3}^{N} P_j$

C. Percent engines requiring calibration after second test = P(i)

Of these, the four "corners" of the bivariate distribution are necessary (see figure 1).

Prior to this analysis, the contractor had been using arbitrary decision limits of nominal $\pm$ (2000 = 2 σ_{RR}). Advantages gained by minimizing expected number of tests are also obtained from A, B, and C above, as follows:

COMPARISON OF DECISION LIMITS

ASSUMING THAT AFTER 9 UNSUCCESSFUL TESTS THE ENGINE IS SCRAPPED

σ_{EE} = 1200 lbs. $\rho = .95$ σ_{RR} = 600 lbs.

(Spec. = Nominal $\pm$ 1.6 Sigma)

(Assume 1 Engine = 40 Tests)	Decision Limit = nominal $\pm$ 0.6 σ_{Xi}		Optimum Dec. Limit = nominal $\pm$ 1.7σ_{Xi}
• Prob. of Acceptance after 3 Tests	.45		~~.87~~ .84
• % Engines requiring calibration after 2nd test	55% (of these, 20% are necessary)		~~10%~~ 14% (of these, ~~77%~~ 91% are necessary)
• Average number of tests required for acceptance	4.11 (due to recalibration)		3.20
	0.61 (due to scrapped engine)		~~0.08~~ 0.12
	4.72 (Total)		~~3.26~~ 3.32
	Δ = 1.5 tests/engine		
• Expected Number of Scrapped Engines Per 100 Tested	1.5		0.29
• % Engines Accepted after N Tests	45.1	N = 3	~~86.7~~ 84.1
	69.9	N = 4	~~95.5~~ 96.0
	83.5	N = 5	~~99.0~~ 98.95
	90.9	N = 6	~~99.6~~ 99.54
	95.0	N = 7	~~99.76~~ 99.67
	97.3	N = 8	~~99.79~~ 99.70
	98.5	N = 9	~~99.80~~ 99.71

COMPARISON OF DECISION LIMITS

ASSUMING THAT AFTER 9 UNSUCCESSFUL TESTS THE ENGINE IS SCRAPPED

σ_{EE} = 1500 lbs. $\rho = .965$ σ_{RR} = 600 lbs.

(Spec. = Nominal $\pm$ 1.3 Sigma)

(Assume 1 Engine = 40 Tests)	Decision Limit = nominal $\pm$ 0.5 σ_{Xi}		Optimum Dec. Limit = nominal $\pm$ 1.5 σ_{X_i}
Prob. of Acceptance after 3 Tests.	.38		~~.78~~ .70
%Engines Requiring Calibration After 2nd Test	62% (of these, 31% are necessary)		~~16%~~ 28% (of these, 84% are necessary)
Average Number of Tests Required for Acceptance	4.4	(due to recalibration)	~~3.3~~ 3.42
	1.4	(due to scrapped engine)	~~0.5~~ 0.19
	5.8	(Total)	~~3.8~~ 3.61

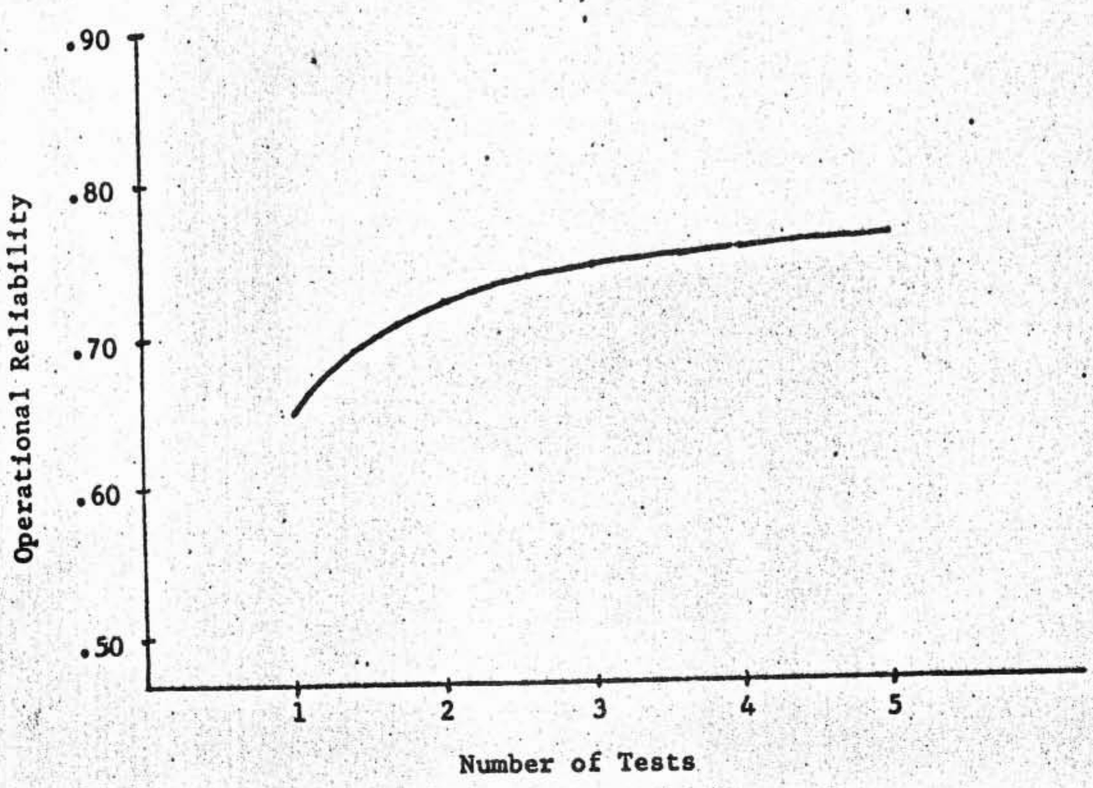
Δ = 2.2 Tests/ Engine

Expected Number of Scrapped Engines per 100 Tested	3.5		~~1.2~~ 0.47
% Engines Accepted after N Tests	38.0	N = 3	~~78.0~~ 70.4
	61.9	N = 4	~~90.6~~ 90.5
	76.4	N = 5	~~96.5~~ 96.7
	85.4	N = 6	~~98.0~~ 98.7
	91.0	N = 7	~~98.5~~ 99.3
	94.4	N = 8	~~98.7~~ 99.47
	96.5	N = 9	~~98.8~~ 99.53

MODEL 2: Reliability ≤ 1.0; Engine is scrapped after 5 unsuccessful tests. Assume that the engine is scrapped when the contractor fails to meet requirements for acceptance after 5 successive tests with calibration. Let $1\text{-}R_1$ be the probability of failure in the first test, where "failure" is any event that causes a single additional test ~~and calibration~~ as in table 2, and similarly for $1\text{-}R_2$ in the second test, etc. A curve of reliability vs. number of tests may be obtained from past experience, as in figure 2.

Figure 2

OPERATIONAL RELIABILITY VS. NUMBER OF TESTS



Let the notation "$1_c\ 2_{F2}\ 3_c\ 4\ \bar{5}$ describe the following event:

Calibration after first test.

Failure during second test.

Calibration after third test.

Thrust in fourth test within decision limits.

Average thrust of fourth and fifth tests within specification limits.

As before, the engine is always calibrated after the first test, unless failure occurs. Using the notation $P(i)$ and $P(\bar{i})$ as in model 1, probabilities for the various events are as follows:

TABLE 2

EVENT	PROBABILITY
$1_c\ 2\ \bar{3}$	$R_1\ R_2\ R_3\left[1 - P(i) - P(\bar{i})\right]$
$1_c\ 2_c\ 3\ \bar{4}$	$R_1\ R_2\ R_3\ R_4\ P(i)\left[1 - P(i) - P(\bar{i})\right]$
$1_{F1} 2_c\ 3\ \bar{4}$	$(1 - R_1)R_2\ R_3\ R_4\left[1 - P(i) - P(\bar{i})\right]$
$1_c\ 2_{F2} 3\ \bar{4}$	$(1 - R_2)R_1\ R_3\ R_4\left[1 - P(i) - P(\bar{i})\right]$
$1_c\ 2_c\ 3_c\ 4\ \bar{5}$	$R_1\ R_2\ R_3\ R_4\ R_5\left[P(i)\right]^2\left[1 - P(i) - P(\bar{i})\right]$
$1_c\ 2\ \bar{3}_c\ 4\ \bar{5}$	$R_1\ R_2\ R_3\ R_4\ R_5\left[1 - P(i)\right]\left[P(\bar{i})\right]\left[1 - P(i) - P(\bar{i})\right]$
$1_{F1} 2_{F2} 3_c\ 4\ \bar{5}$	$(1 - R_1)(1 - R_2)\ R_3\ R_4\ R_5\left[1 - P(i) - P(\bar{i})\right]$
$1_{F1} 2_c\ 3_{F3} 4\ \bar{5}$	$(1 - R_1)(1 - R_3)\ R_2\ R_4\ R_5\left[1 - P(i) - P(\bar{i})\right]$
$1_c\ 2_{F2} 3_{F3} 4\ \bar{5}$	$(1 - R_2)(1 - R_3)\ R_1\ R_4\ R_5\left[1 - P(i) - P(\bar{i})\right]$
$1_{F1} 2_c\ 3_c\ 4\ \bar{5}$	$(1 - R_1)\ R_2\ R_3\ R_4\ R_5\left[P(i)\right]^2\left[1 - P(i) - P(\bar{i})\right]$
$1_c\ 2_{F2} 3_c\ 4\ \bar{5}$	$(1 - R_2)\ R_1\ R_3\ R_4\ R_5\ P(i)\ \left[1 - P(i) - P(\bar{i})\right]$
$1_c\ 2_c\ 3_{F3} 4\ \bar{5}$	$(1 - R_3)\ R_1\ R_2\ R_4\ R_5\ P(i)\ \left[1 - P(i) - P(\bar{i})\right]$

Assuming that the cost of one engine is equivalent to the cost of M tests, and letting P_j be the sum of probabilities in table 2 associated with j tests, the expected number of tests per accepted engine is given by equation (4).

Case 1: Reliability < 1.0

In illustration suppose $\sigma_{EE} = 1200$ lbs., $\sigma_{RR} = 600$ lbs., specification limits are nominal $\pm$ 2000 lbs., and the cost of one engine is equavalent to the cost of 40 tests. Then as in model 1, case 1, we have:

$$\sigma_{X_i} = 1340 \text{ lbs.}$$

$$\rho_{X_i, (X_i + X_{i+1})/2} = .95$$

$$\sigma_{(X_i+X_{i+1})/2} = 1270 \text{ lbs}$$

Number of standard deviations between nominal and specification limit = 1.57.

Calculate P_j from table 2 for j = 3, 4, 5, utilizing operational reliability values of figure 2. By trial and error, equation (4) is minimized when the decision limits are nominal $\pm$ 1.8 standard deviations, and E(N) = 24.6 tests per accepted engine.

Case 2: Reliability = 1.0 (Same correlation coefficient and standard deviations as in case 1)

It is of interest to observe the partial effect of reliability on the optimum decision limits and expected number of tests, E(N). Let R_1 through R_5 be 1.0. Then utilizing table 2, (or table 1 for j = 3, 4, 5) calculate P_j. The standard deviations of X_i and $(X_i+X_{i+1})/2$ and correlation coefficient are the same as in case 1. Equation (4) is minimized when the decision limits are nominal $\pm$ 1.5 standard deviations and E(N) = 3.6 tests per accepted engine. In comparing these values to those in case 1, note that the optimum decision limits become tighter, and the number of tests per accepted engine decreases as reliability increases.

By comparison of results in Model 1, case 1 to those of Model 2, case 2, a measure of the effect of scrapping the engine after 9 versus 5 tests is obtained. The optimum decision limits are nominal $\pm$ 1.7 standard deviations in the former, and E(N) = 3.3 tests per accepted engine; in the latter, the optimum decision limits are nominal $\pm$ 1.5 standard deviations, and E(N) = 3.6 tests.

APPLICATIONS:

The minimum expected number of tests per accepted engine, E(N), provides a convenient yardstick for trade-off studies. For example, one might want to determine whether or not the cost of overhauling test facilities in order to improve operational reliability by, say, 5%, is worthwhile. Or, one might want to determine whether the cost of reducing engine-to-engine variability by improving calibration techniques or equipment is offset by the reduced number of tests required for acceptance, etc.

REFERENCES:

(1) Tables of the Bivariate Normal Distribution Function and Related Functions, National Bureau of Standards, U. S. Department of Commerce

(2) Tables of Normal Probability Functions, National Bureau of Standards, U. S. Department of Commerce.

APPENDIX

$$\rho_{x_i, \frac{x_i + x_{i+1}}{2}} = \frac{\operatorname{cov}\left(x_i, \frac{x_i + x_{i+1}}{2}\right)}{\sigma_{x_i}\ \sigma_{\frac{x_i + x_{i+1}}{2}}}$$

Substituting from equations (7) and (8), this is

$$= \frac{\sigma_{x_i}^2 + \rho_{x_i, x_{i+1}} \sigma_{x_i} \sigma_{x_{i+1}}}{\sigma_{x_i}\sqrt{\sigma_{x_i}^2 + \sigma_{x_{i+1}}^2 + 2\rho_{x_i, x_{i+1}} \sigma_{x_i} \sigma_{x_{i+1}}}} \qquad (5)$$

Substituting from equation (9),

$$= \frac{\sigma_{x_i}^2 + \frac{1}{2}(\sigma_{x_i}^2 + \sigma_{x_{i+1}}^2 - 2\sigma_{RR}^2)}{\sigma_{x_i}\sqrt{2\sigma_{x_i}^2 + 2\sigma_{x_{i+1}}^2 - 2\sigma_{RR}^2}}$$

$$= \frac{3\sigma_{x_i}^2 + \sigma_{x_{i+1}}^2 - 2\sigma_{RR}^2}{\sigma_{x_i}\sqrt{8(\sigma_{x_i}^2 + \sigma_{x_{i+1}}^2 - \sigma_{RR}^2)}} \qquad (6)$$

assuming $\sigma_{x_i} = \sigma_{x_{(i+1)}}$,

$$\rho_{x_i, \frac{x_i + x_{i+1}}{2}} = \frac{2\sigma_{x_i}^2 - \sigma_{RR}^2}{\sigma_{x_i}\sqrt{2(2\sigma_{x_i}^2 - \sigma_{RR}^2)}}$$

$$= \sqrt{1 - \frac{\sigma_{RR}^2}{2\sigma_{x_i}^2}} \qquad (6A)$$

or, letting $\sigma_{x_i} = \sigma_{x_{i+1}}$ in equation (5), this result may be expressed as

$$\rho_{x_i, \frac{x_i + x_{i+1}}{2}} = \frac{1}{\sqrt{2}}\sqrt{1 + \rho_{x_i, x_{i+1}}} \qquad (6B)$$

Let σ_{EE}^2 be the engine-to-engine variance component. Using the relation $\sigma_{x_i} = \sqrt{\sigma_{EE}^2 + \sigma_{RR}^2}$ together with equations (6A) and (8B), the volume in the corners of figure 1 are computed from a bivariate normal distribution table. Using this result plus a univariate normal table, $P(i)$ and $P(\bar{i})$ as defined pgs. 4 and 5, are calculated.

$$\operatorname{cov}\left(X_i, \frac{X_i + X_{i+1}}{2}\right)$$

$$= E\left[X_i\left(\frac{X_i + X_{i+1}}{2}\right)\right] - E(X_i)E\left(\frac{X_i + X_{i+1}}{2}\right)$$

$$= \frac{1}{2}\left\{E(X_i^2) + E(X_i X_{i+1}) - [E(X_i)]^2 - E(X_i)E(X_{i+1})\right\}$$

$$= \frac{1}{2}\left[\sigma_{X_i}^2 + \operatorname{cov}(X_i, X_{i+1})\right]$$

$$= \frac{1}{2}\left[\sigma_{X_i}^2 + \rho_{X_i, X_{i+1}}\sigma_{X_i}\sigma_{X_{i+1}}\right] \qquad (7)$$

If $\sigma_{X_i} = \sigma_{X_{i+1}}$, this becomes

$$\operatorname{cov}\left(X_i, \frac{X_i + X_{i+1}}{2}\right) = \frac{\sigma_{X_i}^2}{2}\left[1 + \rho_{X_i, X_{i+1}}\right] \qquad (7A)$$

$$\sigma_{\frac{X_i + X_{i+1}}{2}} = \frac{1}{2}\sqrt{\sigma_{X_i}^2 + \sigma_{X_{i+1}}^2 + 2\rho_{X_i, X_{i+1}}\sigma_{X_i}\sigma_{X_{i+1}}} \qquad (8)$$

If $\sigma_{X_i} = \sigma_{X_{i+1}}$,

$$\sigma_{\frac{X_i + X_{i+1}}{2}} = \frac{\sigma_{X_i}}{\sqrt{2}}\sqrt{1 + \rho_{X_i, X_{i+1}}} \qquad (8A)$$

rrom (6B), this becomes

$$\sigma_{\frac{X_i + X_{i+1}}{2}} = \sigma_{X_i}\,\rho_{X_i, \frac{X_i + X_{i+1}}{2}} \qquad (8B)$$

from equation (9A), (8A) becomes

$$\sigma_{\frac{x_i + x_{i+1}}{2}} = \sqrt{\sigma_i^2 - \frac{\sigma_{RR}^2}{2}} \qquad (8C)$$

Sum of squares run-to-run is

$$SS_{RR} = \frac{1}{2} \sum_{\alpha=1}^{N} (X_{i_\alpha} - X_{i+1_\alpha})^2$$

$$= \frac{1}{2} \sum_{\alpha=1}^{N} \left[(X_{i_\alpha} - \mu_i) - (X_{i+1_\alpha} - \mu_{i+1}) + (\mu_i - \mu_{i+1})\right]^2$$

Assuming $\mu_i = \mu_{i+1}$, and since $MS_{RR} = \frac{SS_{RR}}{N}$,

$$MS_{RR} = \frac{1}{2}\left[\frac{\Sigma (X_i - \mu_i)^2}{N} + \frac{\Sigma (X_{i+1} - \mu_{i+1})^2}{N} - \frac{2\,\Sigma (X_i - \mu_i)(X_{i+1} - \mu_{i+1})}{N}\right].$$

Taking expectations, the run-to-run variance component is:

$$\sigma_{RR}^2 = \frac{1}{2}\left[\sigma_{x_i}^2 + \sigma_{x_{i+1}}^2 - 2\,\mathrm{cov}(x_i, x_{i+1})\right]$$

$$= \frac{1}{2}\left[\sigma_{x_i}^2 + \sigma_{x_{i+1}}^2 - 2\rho_{x_i, x_{i+1}} \sigma_{x_i} \sigma_{x_{i+1}}\right]$$

and

$$\rho_{x_i, x_{i+1}} = \frac{\sigma_{x_i}^2 + \sigma_{x_{i+1}}^2 - 2\,\sigma_{RR}^2}{2\,\sigma_{x_i}\sigma_{x_{i+1}}} \qquad (9)$$

If $\sigma_{x_i} = \sigma_{x_{i+1}}$

$$\rho_{x_i, x_{i+1}} = 1 - \frac{\sigma_{RR}^2}{\sigma_{x_i}^2} \qquad (9A)$$

LIMITS AND RELATIONSHIPS:

$$\rho_{x_i, x_{i+1}} = 2\rho^2_{x_i, \frac{x_i + x_{i+1}}{2}} - 1 = 1 - \frac{\sigma_{RR}^2}{\sigma_{x_i}^2}$$

When $\sigma_{EE} = 0$, $\rho_{x_i, x_{i+1}} = 0$; when $\sigma_{RR} = 0$, $\rho = 1.0$

$\rho_{x_i, \frac{x_i + x_{i+1}}{2}}$	$\rho_{x_i, x_{i+1}}$	$\frac{\sigma_{RR}^2}{\sigma_{x_i}^2}$	$\frac{\sigma_{RR}}{\sigma_{x_i}}$
1.00	1.00	0	0
.99	.96	.04	.20
.98	.92	.08	.28
.97	.88	.12	.35
.96	.84	.16	.40
.95	.80	.20	.45
.90	.62	.38	.62
.85	.44	.56	.75
.80	.28	.72	.85
.75	.12	.88	.94
$1/\sqrt{2}$ = .707	0	1.00	1.00

FIGURE 3

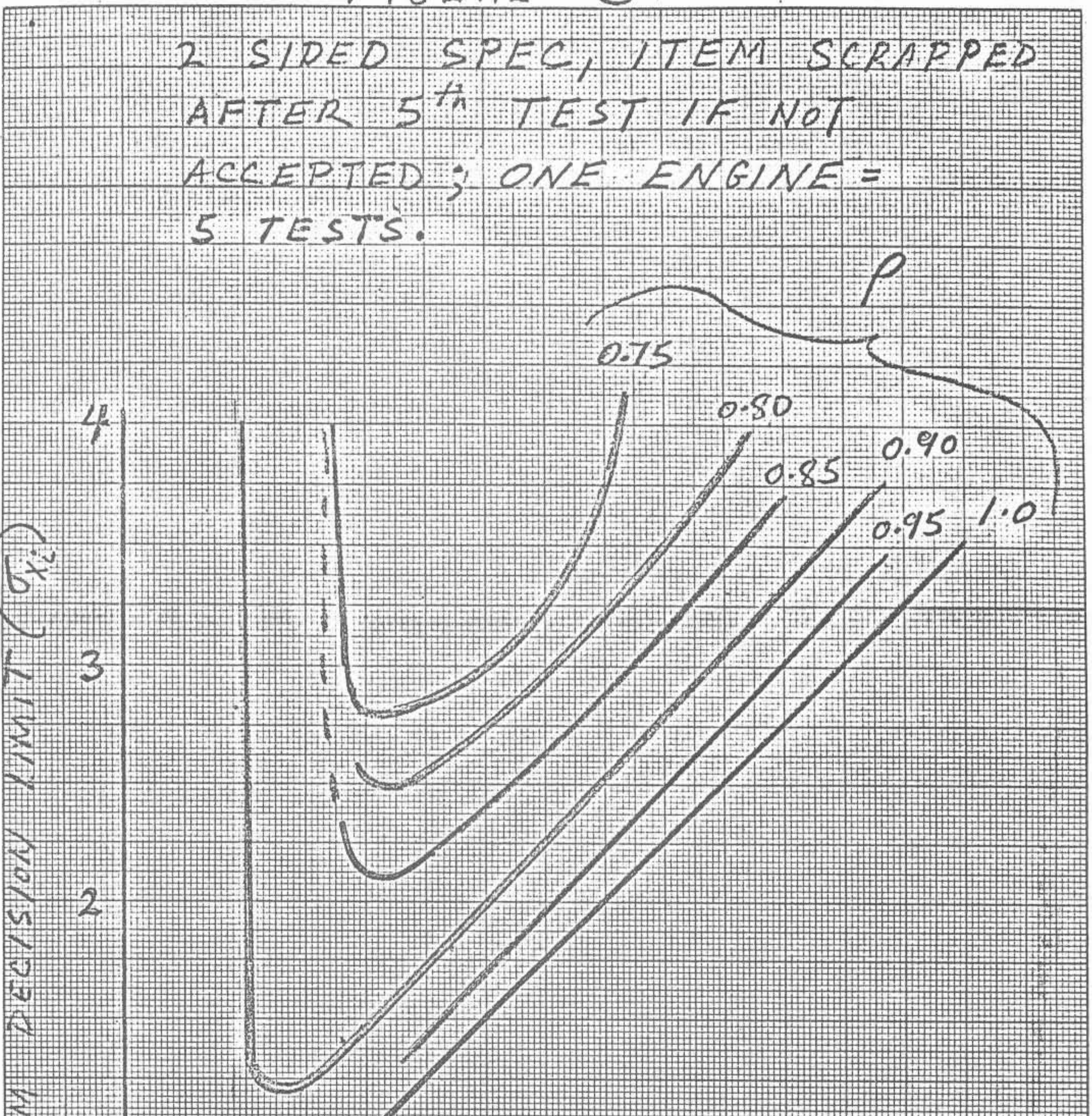

2 SIDED SPEC, ITEM SCRAPPED
AFTER 5th TEST IF NOT
ACCEPTED; ONE ENGINE =
5 TESTS.
ρ
0.75
0.80
0.90
0.85
0.95
1.0
4
3
2
1
0
0
1
2
3
OPTIMUM DECISION LIMIT (σ_{x_i})
SPEC. LIMIT, (σ_{x_i})

FIGURE 4

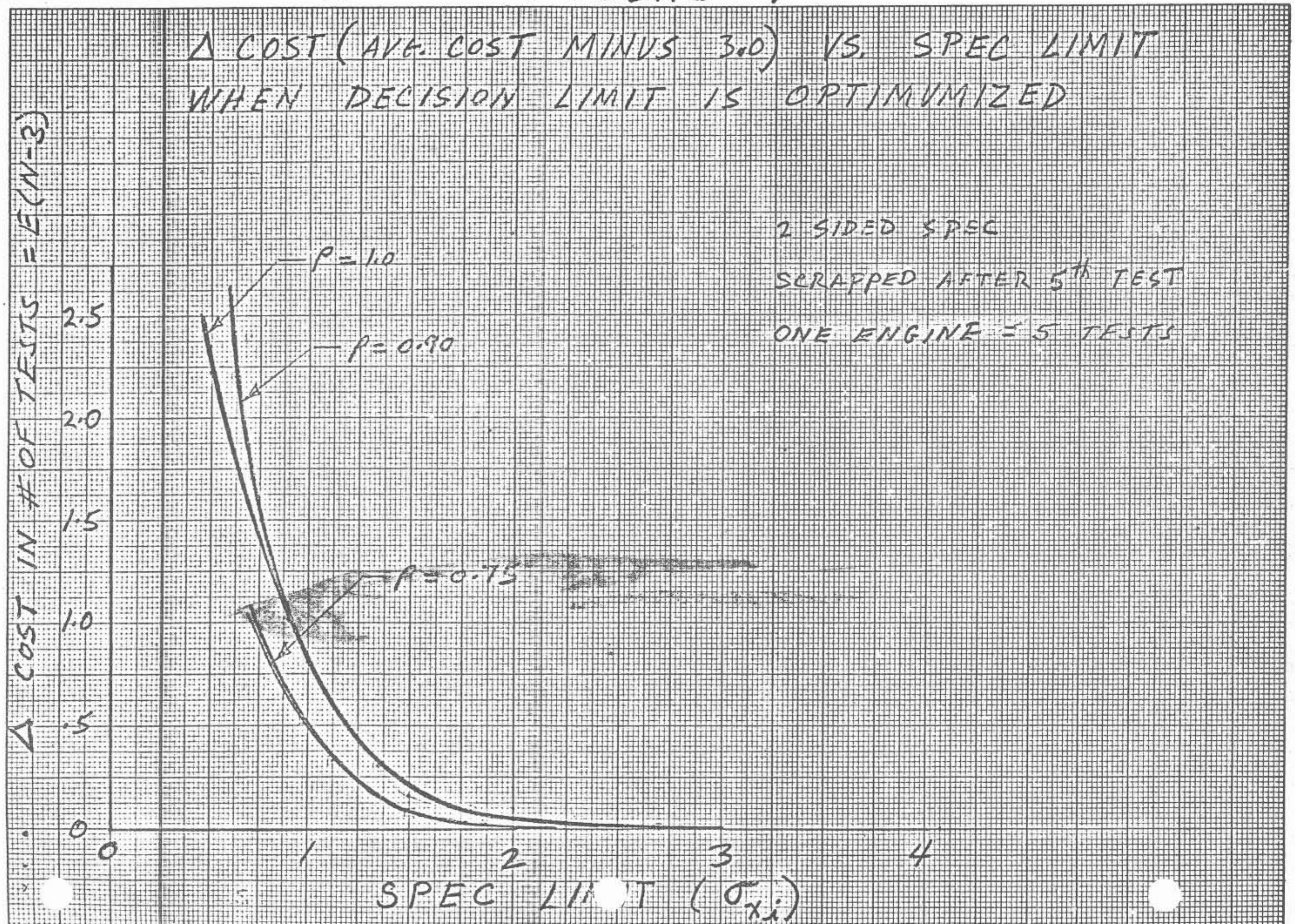
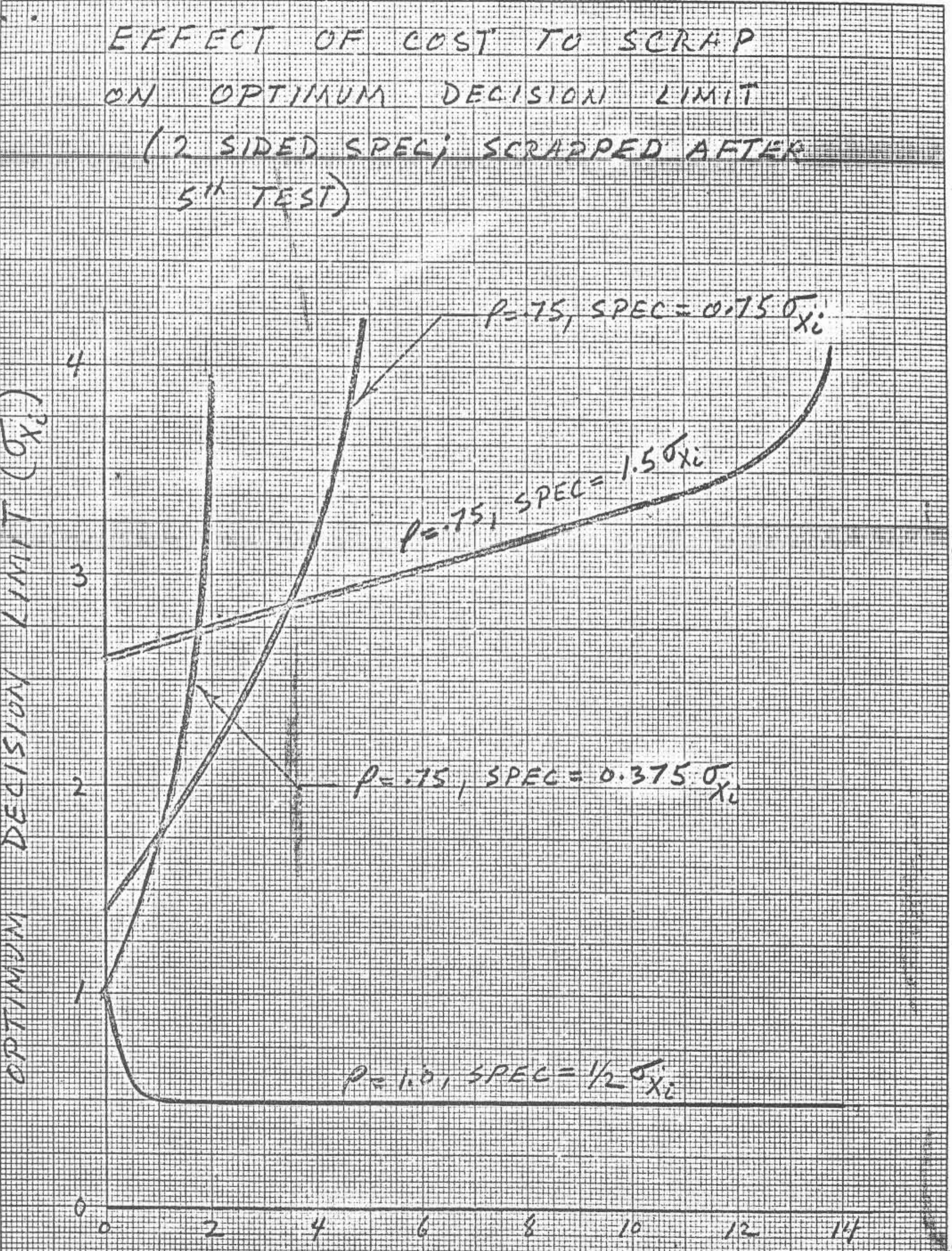
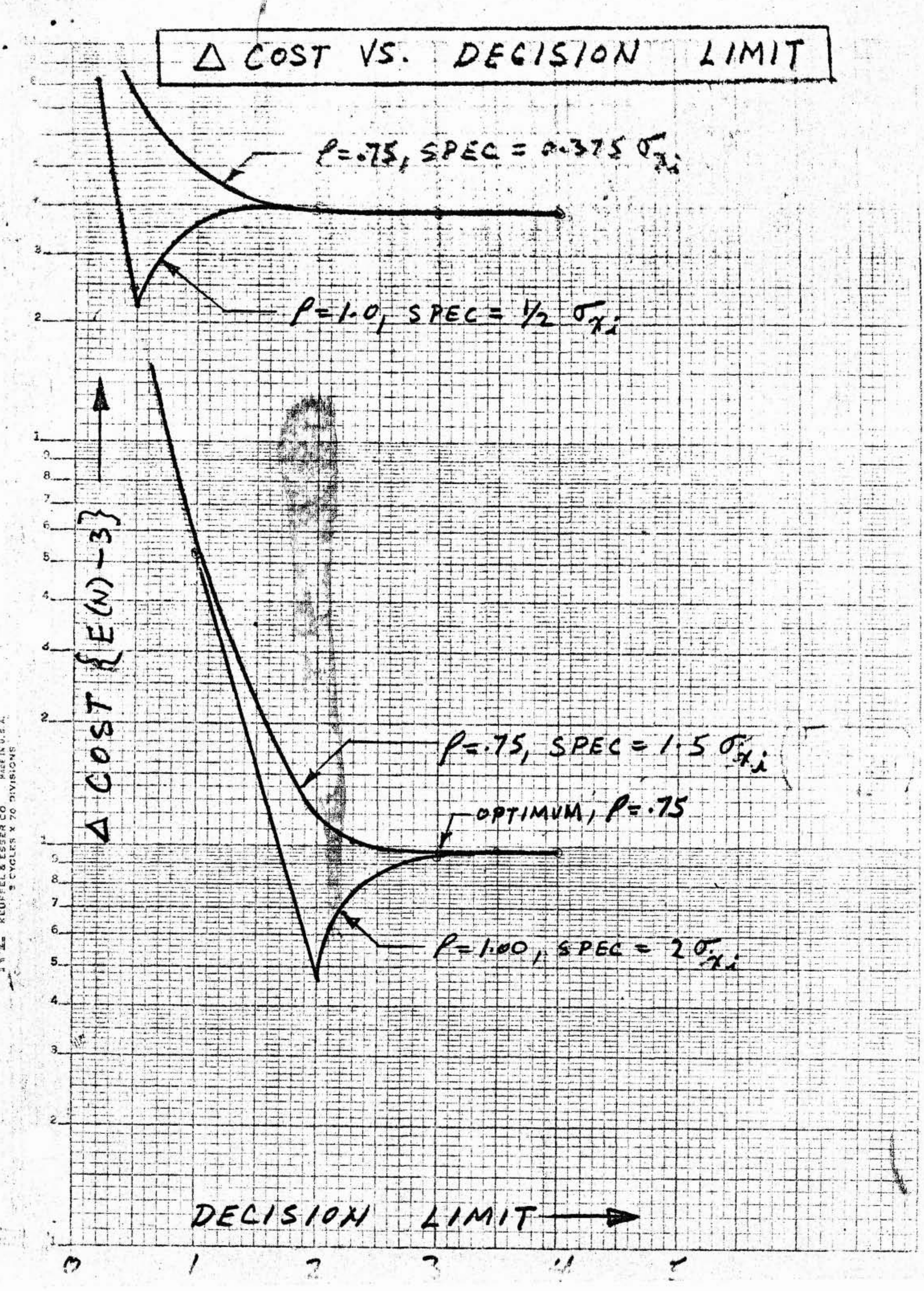

Δ COST (AVG. COST MINUS 3.0) VS. SPEC LIMIT
WHEN DECISION LIMIT IS OPTIMIMIZED
2 SIDED SPEC
SCRAPPED AFTER 5th TEST
ONE ENGINE = 5 TESTS
ρ = 1.0
ρ = 0.90
ρ = 0.75
Δ COST IN # OF TESTS = E(N-3)
0
.5
1.0
1.5
2.0
2.5
0
1
2
3
4
SPEC LIMIT (σxi)

FIGURE 5



EFFECT OF COST TO SCRAP
ON OPTIMUM DECISION LIMIT
(2 SIDED SPEC; SCRAPPED AFTER 5th TEST)
ρ=.75, SPEC=0.75 σXi
ρ=.75, SPEC=1.5 σXi
ρ=.75, SPEC=0.375 σXi
ρ=1.0, SPEC=½ σXi
OPTIMUM DECISION LIMIT (σXi)
M = COST TO SCRAP IN # OF TESTS
0
1
2
3
4
0
2
4
6
8
10
12
14
KEUFFEL & ESSER CO.



Δ COST VS. DECISION LIMIT
ρ=.75, SPEC = 0.375 σ
ρ=1.0, SPEC = ½ σ
ρ=.75, SPEC = 1.5 σ
OPTIMUM, ρ=.75
ρ=1.00, SPEC = 2σ
Δ COST {E(N)−3}
DECISION LIMIT
0
1
2
3
4
5
SEMI-LOGARITHMIC
KEUFFEL & ESSER CO.
359-71