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CONVECTIVE ENERGY TRANSPORT IN STELLAR ATMOSPHERES. I
A CONVECTIVE CELL MODEL*

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ABSTRACT

The motion in a convectively unstable region is expanded into an ensemble of convective cells. Each of these cells interacts with the surrounding medium according to the semi-empirical model proposed by Turner (1963). Possible detailed models of the flow patterns within each cell are presented. The radius and velocity of these cells are given as functions of distance moved. The convective flux and rms velocity are given as averages over the ensemble of cells. As in the standard mixing length theory the principle uncertainty remains the average initial radius of the cells.

I. INTRODUCTION

Many problems in astrophysics require the knowledge of convective energy fluxes and convective velocities. At present all applied theoretical treatments of such problems make use of the mixing length theory. This type of theory was first proposed by Prandtl (1932) for use in treating meteorological convection. Siedentopf (1933a,b, 1935) and Biermann (1937, 1942) then applied the mixing length theory to the study of stellar atmospheres and interiors. Important improvements were later added to this theory by Vitense (1953) and Böhm-Vitense (1958). This last work gives the mixing length theory as is used in most recent studies of stellar interiors and atmospheres.

In contrast, most theoretical descriptions of convection start from a modal analysis of the equations of motion. In principle this approach requires that the linearized equations of motion be solved as an eigenvalue problem. The resulting eigenmodes must then be coupled through the non-linear terms in the equation of motion $[(\underline{v} \cdot \text{grad})\, \underline{v}$ and $(\underline{v} \cdot \text{grad}) T']$, some form of statistical cloture approximation must be applied and finally the average amplitudes derived to give the convective flux. Some progress has been made on individual parts of this program; however, the varying degree of approximation used in the different parts renders the results incompatible. Böhm (1963a,b) has carried out the first stage of the calculation with very few approximations, but made no attempt at the remaining stages. In fact, it is doubtful that the subsequent steps can be carried out at all, since it has never been shown that Böhm's fourth order differential operator is self-adjoint. If, as Spiegel (1965) has indicated, this operator is not self-adjoint, then the prospects for coupling the non-orthogonal modes are not good.

There have been numerous procedures suggested for mode coupling and averaging: Ledoux, Schwarzschild, and Spiegel (1961) used Heisenberg (1948) turbulent viscosity, Unno (1961) adjusted the turbulent viscosity to set the Reynolds number to 30, Kraichnan (1959a,b, 1964, 1965) used the direct interaction approximation, Edwards (1964) used the random phase approximation, and Herring (1965) used the self-consistent field approximation. None of these treatments used realistic normal modes and all depended strongly on the simple properties of the Fourrier modes.

In view of the limitations on the applicability of the modal analysis theories and the desirability of a theory with the simplicity of the mixing length theory, we shall adopt in this paper a model of the convective motion which is closely akin to the mixing length model. We shall expand the motion into an ensemble of convective cells and follow the history of an individual cell with the full non-linear equations of motion. The convective velocity and flux is then given by an average over this ensemble. In the present work we leave the average initial radius of these cells as an unspecified parameter. The description of the behavior of these cells includes the interaction between the cells and their environment according to the semi-empirical model proposed by Turner (1963). Turner's model involves a tendency for the mass of the convective cell to increase by the entrainment of surrounding matter into the organized cell and a tendency for the mass of the cell to decrease by a process of surface errosion due to random thrusts of matter from the surrounding turbulent medium into the organized flow patterns of the cell. This model thus proposes a superposition of organized cellular motion on a smaller scale, random turbulent velocity field. Sections II, IIIa, and IIIb examine the properties of the organized cells while §§ IIIc, IV, and V examine some aspects of the interrelation of these velocity fields.

II. THE ORGANIZED INTERNAL MOTION OF A CONVECTIVE CELL

A rising cell of hot gas in the earth's atmosphere is known as a convective thermal. Morton, Taylor, and Turner (1956) have described the large scale behavior of these convective cells by using the concept of entrainment, whereby some of the surrounding undisturbed matter is swept into the cell and mixed with the matter inside. This convective thermal model was applied by Turner (1963) to the motion of convective cells in turbulent surroundings. He tested his theory through experiments with salt solution convective cells in turbulent pure water and found good agreement. He (Turner 1964a,b) then added some detail to the model by assuming that the internal motion of the cell could be represented by a Hill spherical vortex (Hill 1894; Lamb 1932, Art. 165).

A true convective thermal is produced by the sudden appearance of a small but strong density fluctuation. In a stellar atmosphere there is no source of such strong density fluctuations and a convective cell must be initiated by the chance association of several small fragments which have a net vertical momentum. This vertical motion then produces a small density fluctuation over the entire moving region. This region initially moves as a unit with little organized internal motion. Consequently, Turner's detailed model probably does not accurately represent the motion within a stellar convective cell at all stages of its development. We examine below a family of spherical vorticies which contains as special cases Hill's vortex and a uniformly moving spherical region with no internal motion.

We measure velocities with respect to a stationary coordinate system, but project these velocities onto a spherical coordinate system moving with the convective cell. We take the z axis of the system to be in the vertical direction measured positive inward. Let U be the velocity of the center of the cell and b be its radius. For an axially symmetric flow there always exists a stream function ψ which we take to define the vorticies by

$$\left.\begin{aligned} \psi &= -\frac{U}{2}\,\frac{b^3}{r}\sin^2\theta \qquad \text{for } r > b \\ \text{and} \qquad \psi &= -\frac{U}{2n}\left[n + 3 - 3\left(\frac{r}{b}\right)^n\right] r^2 \sin^2\theta \qquad \text{for } r < b \end{aligned}\right\} \tag{1}$$

Velocities in a spherical coordinate system are related to the stream function by

$$v_r = -\frac{1}{r^2 \sin\theta}\left(\frac{\partial\psi}{\partial\theta}\right)_r \tag{2}$$

$$v_\theta = \frac{1}{r \sin\theta}\left(\frac{\partial\psi}{\partial r}\right)_\theta \quad . \tag{3}$$

The velocities for this family of vortices are

$$\left.\begin{aligned} v_r &= \frac{Ub^3}{r^3}\cos\theta \\ v_\theta &= \frac{Ub^3}{2r^3}\sin\theta \end{aligned}\right\} \text{for } r > b \tag{4}$$

and

$$\left.\begin{aligned} v_r &= \frac{U}{n}\left[n + 3 - 3\left(\frac{r}{b}\right)^n\right]\cos\theta \\ v_\theta &= -\frac{U}{n}\left[n + 3 - 3\left(1 + \frac{n}{2}\right)\left(\frac{r}{b}\right)^n\right]\sin\theta \end{aligned}\right\} \text{for } r < b. \tag{5}$$

When n = 2 the vortex is just Hill's vortex which moves at constant velocity when bouyancy forces are absent. In the limit $n \to \infty$, the velocity for $r < b$ just becomes $U\hat{z}$ where $\hat{z}$ is the unit vector in the vertical direction. In this limit there is a discontinuity in v_θ at $r = b$.

These vortices change with time unless $n = 2$ and $\text{curl}\,(g\rho'\hat{z}) = 0$ where ρ' and g are the density fluctuation and acceleration of gravity respectively. When there is a discontinuity or sharp gradient in ρ' near the edge of the convective cell then $\text{curl}\,(g\rho'\hat{z}) \approx g\rho'_0 \sin\theta\, \delta(r-b)\,\hat{\varphi}$ where ρ'_0 is the relatively smooth interior value of ρ' and $\hat{\varphi}$ is the unit vector in the azmuthal direction. There is thus a tendency to produce vorticity ω_φ near the boundary of the convective cell. The velocities in equations (5) give

$$\omega_\varphi = \frac{3U}{2}(n+3)\frac{r^{n-1}}{b^n}\sin\theta \qquad (6)$$

which in the limit of large n is primarily confined to the region just inside of $r = b$ and matches the θ dependence of $\text{curl}\,(g\rho'\hat{z})$.

The buoyancy forces continually tend to produce a vortex with a large n while the inertial forces tend to produce a vortex with n = 2. The precise result of these competing tendencies cannot be obtained without a detailed solution of the equations of motion; however, n can reasonably be expected to be somewhat greater than 2.

The nature of these vortices is most clearly illustrated by plotting lines of constant ψ for velocities measured with respect to the center of the cell. Figures 1 and 2 show these streamlines for the cases n = 2 and 8. In each case the interval in ψ between successive streamlines was $0.045Ub^2$. The density of streamlines indicates the velocity of flow relative to the center of the cell and the motion of an individual particle of fluid is along the streamline.

III. THE INTEGRATED PROPERTIES

a) Kinetic Energy and Vertical Impulse

The total kinetic energy associated with the organized motion of a single convective cell of the type discussed above is simply the integral over all space of the kinetic energy density $\frac{1}{2}\rho|v|^2$. The appropriate velocities are those given in equations (4) and (5). Let the total kinetic energy associated with the organized cell be T_{org} and the volume inside $r = b$ be $\mathcal{V}$. The integral of the kinetic energy density gives

$$T_{org} = \frac{\nu''}{2} \rho \mathcal{V} U^2 \tag{7}$$

with

$$\nu'' = \frac{3n + 9}{2n + 3} \quad . \tag{8}$$

Some care is necessary in discussing the total vertical impulse since the surface integral of the pressure fluctuation does not vanish as $r \to \infty$. Since the velocity for $r > b$ is derivable from a velocity potential, the total vertical momentum between any pair of concentric spheres must vanish by Gauss' theorem. Thus we need only examine the momentum for $r < b$ and the pressure fluctuations on the surface $r = b$. Note that the presence of these pressure fluctuations requires that we use the concept of impulse rather than momentum alone. The flow for $r > b$ is the same as the flow around a solid sphere moving in an inviscid incompressible fluid. Therefore the pressure reactance must add $0.5\ \rho \mathcal{V} U$ to the total vertical impulse just as in this simpler problem (Lamb 1932, Art. 92). We now determine the total vertical impulse P_z associated with a single convective cell by integrating the vertical momentum density ρv_z over the region $r < b$ and adding $0.5\ \rho\mathcal{V}U$. We obtain

$$P_z = \nu'\rho\mathcal{V}U \tag{9}$$

with

$$\nu' = \frac{3n + 13}{2n + 6} \tag{10}$$

In a time δt the changes in the total kinetic energy δT_{total} and total vertical impulse are just due to the total bouyancy force $\mathcal{V}g\rho'$. We have then

$$\frac{\delta T_{total}}{T_{org}} = \frac{2}{\nu''}\frac{g\rho'}{U\rho}\,\delta t \tag{11}$$

and

$$\frac{\delta P_z}{P_z} = \frac{1}{\nu'}\frac{g\rho'}{U\rho}\,\delta t \tag{12}$$

The changes in kinetic energy and total impulse of the organized cell are due both to changes in the velocity of the center U and to changes in the total volume $\mathcal{V}$. Thus

$$\frac{\delta T_{org}}{T_{org}} = \frac{\delta\mathcal{V}}{\mathcal{V}} + 2\,\frac{\delta U}{U} \tag{13}$$

and

$$\frac{\delta P_z}{P_z} = \frac{\delta\mathcal{V}}{\mathcal{V}} + \frac{\delta U}{U} \tag{14}$$

We take the change in velocity of the cell center to be given by equations (14) and (12) and find that there is a net loss of kinetic energy from the organized flow given by

$$\frac{\delta T_{loss}}{T_{org}} = -\frac{\delta \mathcal{V}}{\mathcal{V}} + \frac{2}{\nu'''}\frac{g\rho'}{U\rho}\,\delta t \tag{15}$$

where

$$\frac{1}{\nu'''} = \frac{1}{\nu'} - \frac{1}{\nu''} = \frac{n + 15}{(3n + 13)(3n + 9)} \quad . \tag{16}$$

Equation (15) remains quite accurate even if n is allowed to change with time. Since ρ' and U have the same sign, we see that the volume must always increase rapidly enough to ensure that the right-hand side of equation (15) is negative. If we further suppose that kinetic energy is systematically lost from the organized motion through some process like turbulent viscosity then the volume of the cell must increase more rapidly.

Turner has found empirically that the rate of increase of the volume of a convective cell obeys the equation

$$\frac{d\mathcal{V}}{dt} = \alpha\, 4\pi b^2\, |U| \tag{17}$$

where α is a positive constant in the range 0.25 to 0.333. The relatively rapid increase in the volume of the organized cell given by equation (17) results in mixing substantial amounts of ambient matter with the cell. This process is called entrainment and brings about both a drag effect and a cooling effect. We now use equations (12), (14), and (17) to obtain the equation of motion for the center of the cell:

$$\frac{dU}{dt} = \frac{g\rho'}{\nu'\rho} - \frac{3\alpha|U|}{b}\,U \quad . \tag{18}$$

Let the depth of the center of the cell below an arbitrary zero point in the atmosphere be z. Equation (18) with z as the independent variable is

$$\frac{dU}{dz} = \frac{g\rho'}{\nu'\rho} - \frac{3\alpha|U|}{b} \tag{19}$$

b) The Heat Content of a Convective Cell

We assume in this section that the heat capacity, the opacity, the adiabatic, and the true temperature gradients are constant and examine the total entropy of a convective cell $\rho\mathcal{V}S$. Although the assumption of constant physical parameters is not valid in the layers just below the photosphere, it does apply in the deeper interior. This assumption is made in the usual mixing length theory so that the results of this study can be compared directly to the usual results. In addition we assume that all fluctuations in the thermodynamic quantities are confined to the portion of the cell where $r < b$. Also we make the approximation that the flow is subsonic and assume that the pressure fluctuations implied by the changing velocities discussed in § IIIa do not influence the density or heat content. The rate of change of the entropy of the cell is

$$\frac{d}{dt}(\rho\mathcal{V}S) = -\frac{\rho C_p\mathcal{V}}{T}qT' + \rho S_0\frac{d\mathcal{V}}{dt} \tag{20}$$

where q is the cooling rate of the cell, C_p is the specific heat at constant pressure and S_0 is the average entropy density of the surrounding matter. The final term in equation (20) is simply a result of the fact that the entropy of the material added to the cell by entrainment is not zero. If the cell is optically thin, q is the Newtonian cooling rate while if the cell is optically thick then the heat equation must reduce to an approximate form of the diffusion approximation. Following standard mixing length theory practice (Henyey, Vardya, and Bodenheimer 1965) we take q to be given by the interpolation formula

$$q = \frac{16\ \sigma \kappa T^3}{C_p\ [1 + 4y^2\ (\rho \kappa b)^2]} \tag{21}$$

where κ is the opacity per gram and y is the shape factor discussed by Henyey et al. (1965) relating the temperature gradient at the edge of the cell to the average temperature fluctuation.

Equation (20) can be put in a more usual form by writing the time rate of change of the entropy in terms of the changes in temperature and pressure, and by subtracting the average temperature gradient from both sides. Finally, we note that because of the subsonic velocities the difference $S - S_0$ at each level is to be taken at constant pressure. The heat equation is now

$$\frac{dT'}{dz} = -\frac{T}{H}(\nabla - \nabla_{ad}) - q\frac{T'}{U} - 3\alpha\frac{|U|}{b}\frac{T'}{U} \quad . \tag{22}$$

In equation (22) the variables ∇ and ∇_{ad} are respectively the true and adiabatic logarithmic derivatives of the temperature with respect to pressure. The variable H is the pressure scale height and equation (17) has been used for $(d\mathcal{V}/dt)/\mathcal{V}$. Equations (19) and (22) combine to show that the ratio T'/U is constant provided that ρ'/ρ can be replaced with $-QT'/T$.

Define the following parameters

$$\begin{aligned} v_g &= \left(\frac{gHQ}{\nu'}\right)^{\frac{1}{2}} \\ B &= \frac{qH}{2v_g} \\ \nabla' &= (\nabla - \nabla_{ad} + B^2)^{\frac{1}{2}} - B \end{aligned} \tag{23}$$

The variable ∇' in the absence of entrainment would be the gradient as the cell moves as discussed by Böhm-Vitense (1958). The constant value of T'/U is then

$$\frac{T'}{U} = -\frac{T}{v_g}(\nabla - \nabla')^{\frac{1}{2}} \tag{24}$$

and equations (19) and (22) become

$$\frac{H}{v_g}\frac{dU}{dz} = (\nabla - \nabla')^{\frac{1}{2}} - \frac{3\alpha|U|H}{v_g b} \tag{25}$$

$$\frac{H}{T}\frac{dT'}{dz} = -(\nabla - \nabla') + \frac{3\alpha|U|H}{v_g b}(\nabla - \nabla')^{\frac{1}{2}} \quad . \tag{26}$$

Equations (25) and (24) or (26) define the behavior of U and T' with z when b(z) is known. Note that the process of entrainment reduces both U and T' to the same fraction of their values without entrainment.

c) The Mass of an Organized Cell

The fundamental uncertainties of this theory are the mass of a typical cell and the variation of the mass of a single cell during its motion. At present only partial answers can be provided for these uncertainties. In this section we discuss and amplify Turner's (1963) empirical model for the mass of a convective cell. This model contains no way of choosing the average size of a cell. In a subsequent paper we shall discuss instabilities in these cells which may limit their size.

Turner's convective cell model states that matter is added to the organized motion by the process of entrainment and removed by a process of surface erosion or attrition caused by the ambient smaller scale turbulence.

The effect of this turbulence on the cell can be represented by a disorder front moving into the cell with the ambient rms velocity, V_0. The mass m of the organized cell thus obeys

$$\frac{dm}{dt} = 4\pi b^2 \rho_0 \alpha |U| - 4\pi b^2 \rho V_0 \tag{27}$$

where ρ and ρ_0 are the densities inside and outside the cell respectively. Equation (27) can be written in terms of b when the density ρ is known as a function of time. Equation (26) can be used to define a polytropic exponent γ_c referring to the matter within the cell by

$$\frac{1}{\gamma_c} = \left(\frac{\partial \log \rho}{\partial \log P}\right)_{cell} = \left(\frac{\partial \log \rho}{\partial \log P}\right)_T - Q\nabla_c \tag{28}$$

and

$$\nabla_c = \nabla' + \frac{3\alpha|U|H}{v_g b}(\nabla - \nabla')^{\frac{1}{2}} \quad . \tag{29}$$

We obtain

$$\frac{db}{dz} = \alpha \frac{\rho_0}{\rho}\frac{|U|}{U} - \frac{V_0}{U} - \frac{b}{3\gamma_c H} \quad . \tag{30}$$

For small density fluctuations equation (30) can be further simplified by using

$$\frac{\rho_0}{\rho} \cong 1 + Q\frac{T'}{T} = 1 - \frac{QU(\nabla - \nabla')^{\frac{1}{2}}}{v_g} \tag{31}$$

and by defining

$$\frac{1}{\gamma_e} = \left(\frac{\partial \log \rho}{\partial \log P}\right)_T - Q\nabla' \quad . \tag{32}$$

Equation (30) is then

$$\frac{db}{dz} = \alpha \frac{|U|}{U} - \frac{V_0}{U} - \frac{b}{3\gamma_e H} \tag{33}$$

It should be emphasized here that although the attrition process changes the size of the organized cell, it does not alter the internal momentum density and temperature since the matter removed has the properties of the interior. The final term in equation (33) is of special interest since it introduces the ratio of cell radius to pressure scale height and implies that rising and falling cells behave differently.

IV. THE ENSEMBLE AVERAGE

We now relate V_0 to the rms velocity $\langle U^2\rangle^{\frac{1}{2}}$ of an ensemble of convective cells. As long as the turbulent energy spectrum is reasonably smooth and the scale of the random elements producing attrition does not greatly differ from the scale of the convective cells, we can write

$$V_0 = \beta \langle U^2\rangle^{\frac{1}{2}} \tag{34}$$

where β is a number of order unity. To obtain $\langle U^2\rangle^{\frac{1}{2}}$ we examine a large area A at depth z_0 and time t_0. This area will be occupied by slices of cells whose centers are various distances from z_0. Let ζ be the distance below z_0 of a cell center. The area d^2A occupied by the portions of those cells whose centers originated between the depths $z_1 + \zeta + d\zeta$ and the times t_1 and $t_1 + dt_1$, and whose centers are observed at depth $z_0 + \zeta$ at time t_0 is

$$d^2A = GA\pi\,(b^2 - \zeta^2)\,d\zeta\,dt_1 \tag{35}$$

where G is the number of convective cell centers generated per unit time per unit volume. The radius b and velocity U are functions of z_0 and z_1 alone, hence we can immediately integrate over ζ. Also we write dt_1 as dz_1/U to obtain

$$\frac{dA}{A} = G \frac{4\pi b^3}{3U} dz_1 \quad . \tag{36}$$

The generation rate G is determined by the condition $\int dA = A$ where the integration is over all cells which can contribute to the area at z_0. We now define the ensemble average as

$$\langle U^2 \rangle = \int_{z_{min}}^{z_{max}} wU^2 \, dz_1 \tag{37}$$

where

$$w = \frac{b^3}{N|U|} \tag{38}$$

and

$$N = \int_{z_{min}}^{z_{max}} \frac{b^3}{|U|} dz_1 \quad . \tag{39}$$

The convective flux $\mathcal{F}_c$ is similarly defined as the ensemble average $- \rho C_p \langle UT' \rangle$. By virtue of equation (24) we have

$$\mathcal{F}_c = \frac{\rho C_p T(\nabla - \nabla')^{\frac{1}{2}}}{v_g} \langle U^2 \rangle \quad . \tag{40}$$

An additional condition which must be satisfied in a steady state is

$$\langle U \rangle = 0 \quad . \tag{41}$$

The meaning of this ensemble averaging procedure is that all space in an unstable region is filled with convective cells in various stages of development. It must be emphasized, however, that within this ensemble are slow moving regions composed of fragments of previous cells. These regions represent new cells in the process of formation and contain little or no internal organization. These proto-cells provide a buffer region between fully formed rising and falling cells.

V. THE TYPICAL PROTO-CELL

In this model when a cell is formed it consists of fragments of previous cells. These fragments are part of the velocity field leading to attrition and have an rms velocity of V_0. A proto-cell consisting of N of these fragments will typically have an initial velocity of

$$U_0 = \pm V_0/N^{\frac{1}{2}} \quad . \tag{42}$$

Let the average size of a fragment b_f be related to the initial cell size by

$$b_f = \epsilon \, b_0 \quad . \tag{43}$$

The number of fragments forming a proto-cell is then $N = (b_0/b_f)^3$ where b_0 is the typical proto-cell radius. The average proto-cell therefore has a velocity U_0 given by

$$U_0 = \pm V_0 \, \epsilon^{\frac{3}{2}} \quad . \tag{44}$$

The condition in equation (41) and the differing behavior for rising and falling cells implied by equation (33) requires that there must be a systematic difference between the radius of rising proto-cells and falling proto-cells. We let b_0 be given by

$$b_0 = aH\left(1 + \frac{U_0}{|U_0|}\zeta\right) \tag{45}$$

where ζ is to be determined by applying equation (41) and a is a factor corresponding to the ratio of mixing length to pressure scale height.

VI. NUMERICAL SOLUTIONS

The numerical solutions to the pair of equations (25) and (33) are conveniently given in terms of the dimensionless variables z^*, b^*, and U^* defined by

$$z^* = \frac{z - z_0}{aH}$$

$$b^* = \frac{b}{aH} \tag{46}$$

$$U^* = \frac{U}{av_g(\nabla - \nabla')^{\frac{1}{2}}}$$

where z_0 is the depth at which the cell originates. The variables U_0^*, V_0^*, and b_0^* are similarly defined. Equations (25) and (33) are then after dropping the stars

$$\frac{dU}{dz} = 1 - \frac{3\alpha|U|}{b} \tag{47}$$

and

$$\frac{db}{dz} = \frac{1}{U}(\alpha|U| - V_0) - \chi b \tag{48}$$

where

$$\chi = \frac{a}{3\gamma_e} \quad . \tag{49}$$

The boundary conditions are given by equation (44) and

$$b_0 = 1 + \frac{U_0}{|U_0|} \zeta \quad . \tag{50}$$

In this system the averages $\langle U^2 \rangle$ and $\langle U \rangle$ determine the solution but are also functions of the solution. Consequently, they must be adjusted in an iterative procedure to give self-consistent solutions. The parameter χ can assume a variety of values dependent on the value of a and the physical characteristics of the atmosphere under investigation.

This model of convection involves the parameters α, β, and ϵ. Turner's experiments establish that $\alpha \sim 0.25$-0.35. The parameter ϵ is the ratio of average fragment radius to average cell radius and must be roughly 0.2-0.5. The parameter β is related to ϵ through the turbulent energy spectrum. Let the average kinetic energy, $E(k)$, associated with wave number k be given by

$$E(k) \propto k^{\mu} \quad . \tag{51}$$

We then have

$$\beta = \epsilon^{-\mu/2} \quad . \tag{52}$$

From the theory of isotropic turbulence it is known that near the energy containing peak of the turbulent spectrum, μ is between 1 and -5/3 (Hinze 1959, p. 189).

The important properties of the solutions are ζ, $\langle U^2 \rangle^{\frac{1}{2}}$, z_{max} (the dimensionless distance a cell moves before disappearing). These properites are given in Table 1 for various values of α, ζ, and μ. The mean square velocity is particularly important since it relates the convective flux to the physical properties by

$$\mathcal{F}_c = \langle U^2 \rangle \, a^2 \rho C_P T v_g (\nabla - \nabla')^{\frac{3}{2}} \quad . \tag{52}$$

Finally, Figures 3 and 4 show b, U, and wU^2 as functions of z for two cases.

For certain combinations of the parameters the system of equations has no solution. When the parameter χ is too large, the surface erosion cannot keep up with the expansion of the rising cells and they increase in size without limit. In reality there must be some internal instability which leads to the disintegration of these cells after they have moved a certain distance. We therefore regard the lack of a solution as an indication that the theory is not valid for that particular combination of parameters.

The results of this theory are quite similar to the results of the Böhm-Vitense (B-V) theory. In equation (52) the factor $\langle U^2 \rangle$ replaces the factor 0.5 of the B-V theory. We also have shown that ν is between 1.5 and 2.2. By virtue of equation (24) it is clear that the convective flux must be related to $\langle U^2 \rangle$ rather than $\langle |U| \rangle^2$ as is the case in the B-V theory. Consequently, the turbulent pressure is very closely related to the convective flux and the factor p of the B-V theory must be unity. More important, however, is the fact that this theory provides detailed information about the relative contribution to the convective flux as a function of the distance between the point where a cell originates and the point where it is observed. This contribution function is wU^2 and is shown in Figures 3 and 4. In the calculation of the convective flux in a realistic atmosphere where the physical properties change rapidly, this function must be known.

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TABLE 1

CHARACTERISTICS OF CONVECTIVE CELL SOLUTIONS

α	ϵ	μ	χ	V_0	$\langle U^2 \rangle^{\frac{1}{2}}$	ζ	$\lvert \Delta z \rvert_{max}$ upward motion	$\lvert \Delta z \rvert_{max}$ downward motion
0.25	0.4	0.5	0.00	0.2619	0.2083	0.0000	1.68	1.68
			0.16	0.2635	0.2096	0.0427	1.92	1.48
			0.32	0.2680	0.2131	0.0857	2.24	1.32
0.25	0.4	0.0	0.00	0.2367	0.2367	0.0000	2.18	2.18
			0.16	0.2385	0.2385	0.0574	2.61	1.87
			0.32	0.2432	0.2432	0.1158	3.23	1.63
0.25	0.4	-0.25	0.00	0.2258	0.2532	0.0000	2.53	2.53
			0.16	0.2276	0.2552	0.0675	3.12	2.11
			0.32	0.2323	0.2605	0.1365	4.04	1.84
0.25	0.4	-0.5	0.00	0.2165	0.2721	0.0000	2.92	2.92
			0.16	0.2182	0.2742	0.0799	3.76	2.40
			0.32	0.2224	0.2796	0.1575	5.16	2.04
0.25	0.4	-1.66	0.00	0.1883	0.4029	0.0000	6.57	6.57
			0.16		no solution			
			0.32		no solution			
0.25	0.2	0.0	0.00	0.1698	0.1698	0.0000	3.00	3.00
			0.16	0.1706	0.1706	0.0688	3.87	2.42
			0.32		no solution			
0.333	0.4	0.0	0.00	0.2339	0.2339	0.0000	2.21	2.21
			0.16	0.2350	0.2350	0.0632	2.65	1.89
			0.32	0.2376	0.2376	0.1275	3.28	1.68

REFERENCES

Biermann, L. 1937, Astr. Nachr., 264, 359.

__________ 1942, Zeit. f. Ap., 21, 320.

Böhm, K. H. 1963a, Ap. J., 137, 881.

__________ 1963b, Zeit. f. Ap., 57, 265.

Böhm-Vitense, E. 1958, Zeit. f. Ap., 46, 108.

Edwards, L. F. 1964, J. Fluid Mech., 18, 239.

Heisenberg, W. 1948, Zeit. f. Phys., 124, 628.

Henyey, L. G., Vardya, M. S., and Bodenheimer, P. 1965, Ap. J., 142, 841.

Herring, J. R. 1965, Phys. Fluids, 8, 2219.

Hill, M. J. M. 1894, Roy. Soc. London Phil. Trans., 185, 213.

Hinze, J. 1959, Turbulence (New York: McGraw-Hill Book Co.).

Kraichnan, R. H. 1959a, Phys. Rev., 113, 1181.

__________ 1959b, J. Fluid Mech., 5, 497.

__________ 1964, Phys. Fluids, 1, 1723.

__________ 1965, Phys. Fluids, 8, 575.

Lamb, Sir H. 1932, Hydrodynamics (6th ed., Cambridge: Cambridge University Press).

Ledoux, P., Schwarzschild, M., and Spiegel, E. A. 1961, Ap. J., 133, 184.

Morton, B. R., Taylor, G. I., and Turner, J. S. 1956, Proc. Roy. Soc. (London) 234A, 1.

Prandtl, L. 1932, Beitrage Phys. freien atmos., 19, 188.

Siedentopf, H. 1933a, Astr. Nachr., 247, 297.

__________ 1933b, Astr. Nachr., 249, 53.

__________ 1935, Astr. Nachr., 255, 157.

__________ 1965, Ap. J., 141, 1068.

Spiegel, E. A. 1965, Ap. J., 141, 1068.

Turner, J. S. 1963, J. Fluid Mech., 16, 1.

__________ 1964a, J. Fluid Mech., 18, 195.

__________ 1964b, J. Fluid Mech., 19, 481.

Unno, W. 1961, P.A.S.J., 13, 276.

Vitense, E. 1953, Zeit. f. Ap., 32, 135.

FIGURE CAPTIONS

Fig. 1. Streamlines relative to the center of the cell for $n = 2$.

Fig. 2. Streamlines relative to the center of the cell for $n = 8$.

Fig. 3. wU^2, b, and $|U|$ as functions of Δz for convective cell solutions when $\alpha = 0.25$, $\epsilon = 0.4$, $\mu = -0.5$. Values of χ for all the functions are labeled on wU^2.

Fig. 4. wU^2, b, and $|U|$ as functions of Δz for convective cell solutions when $\alpha = 0.25$, $\epsilon = 0.4$, $\mu = 0.0$. Values of χ for all the functions are labeled on wU^2.

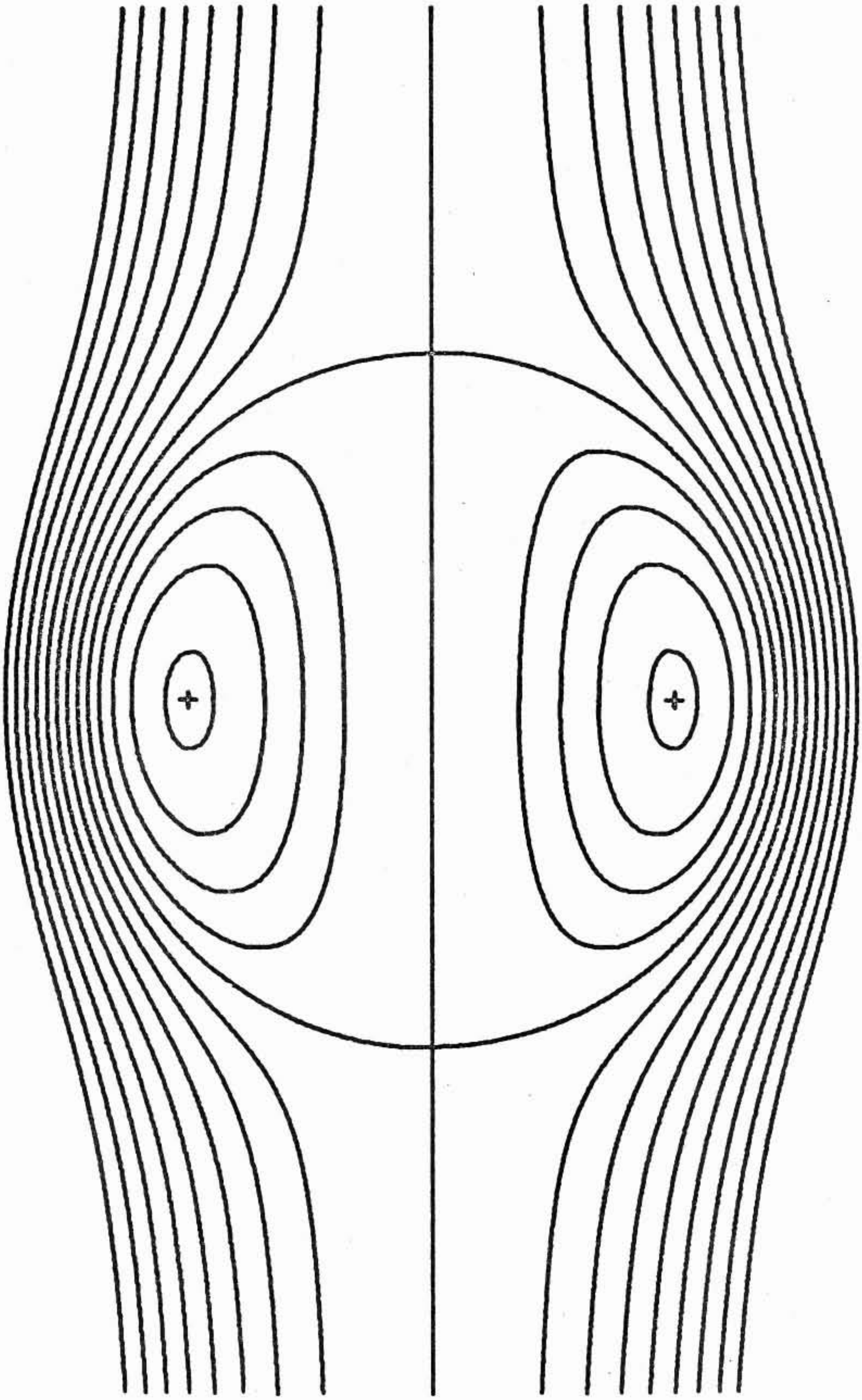

Fig. 1

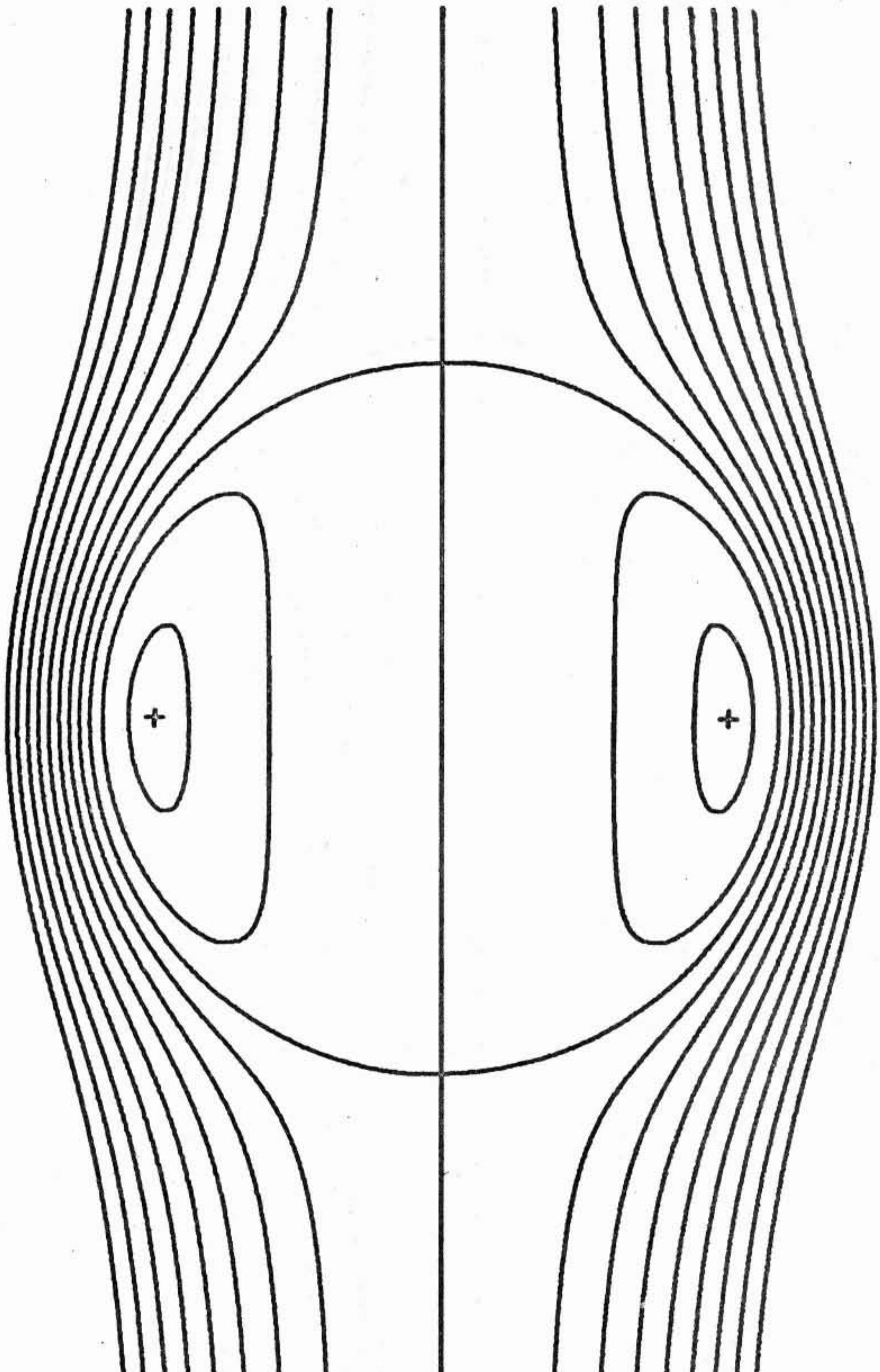

Fig. 2

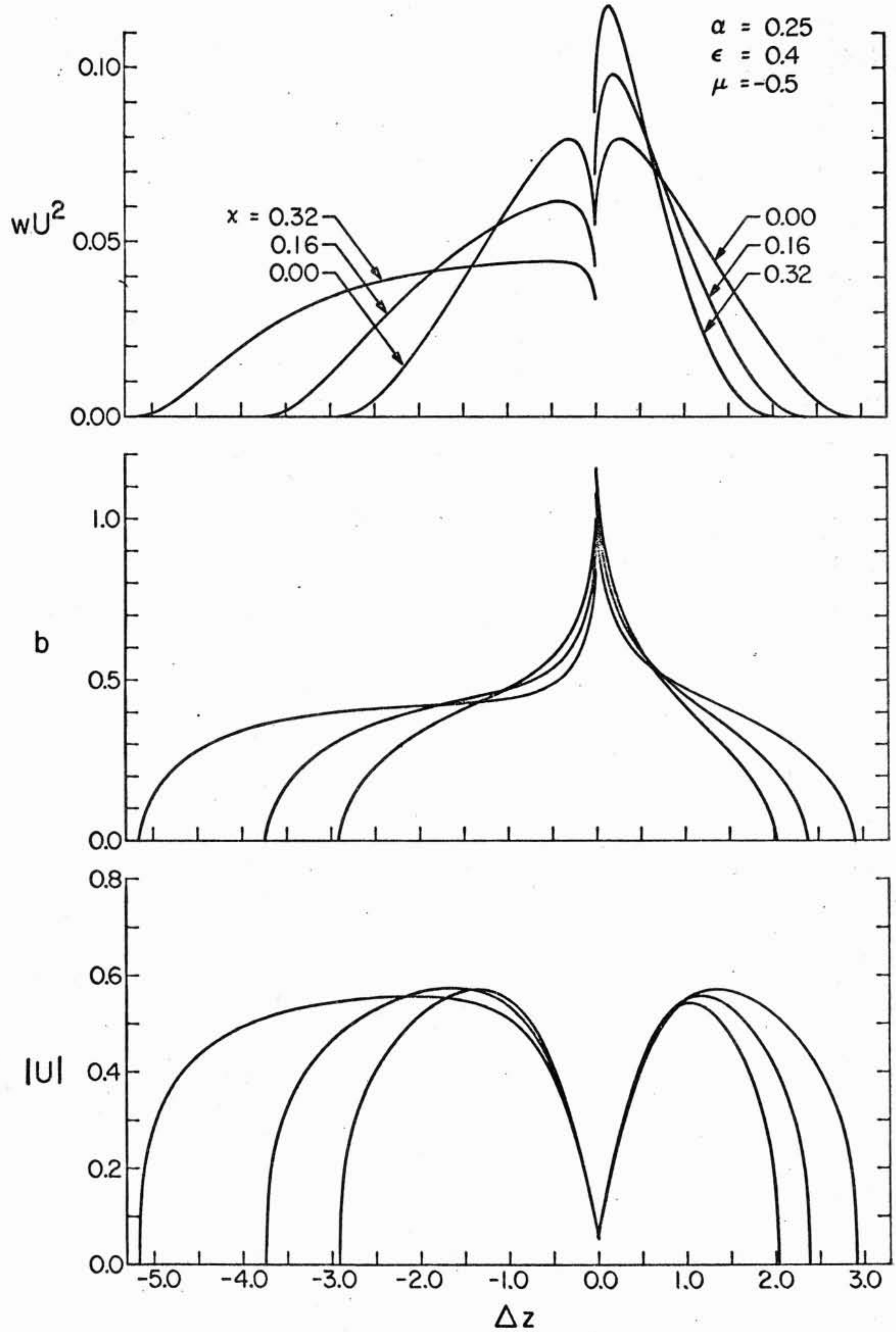

Fig. 3

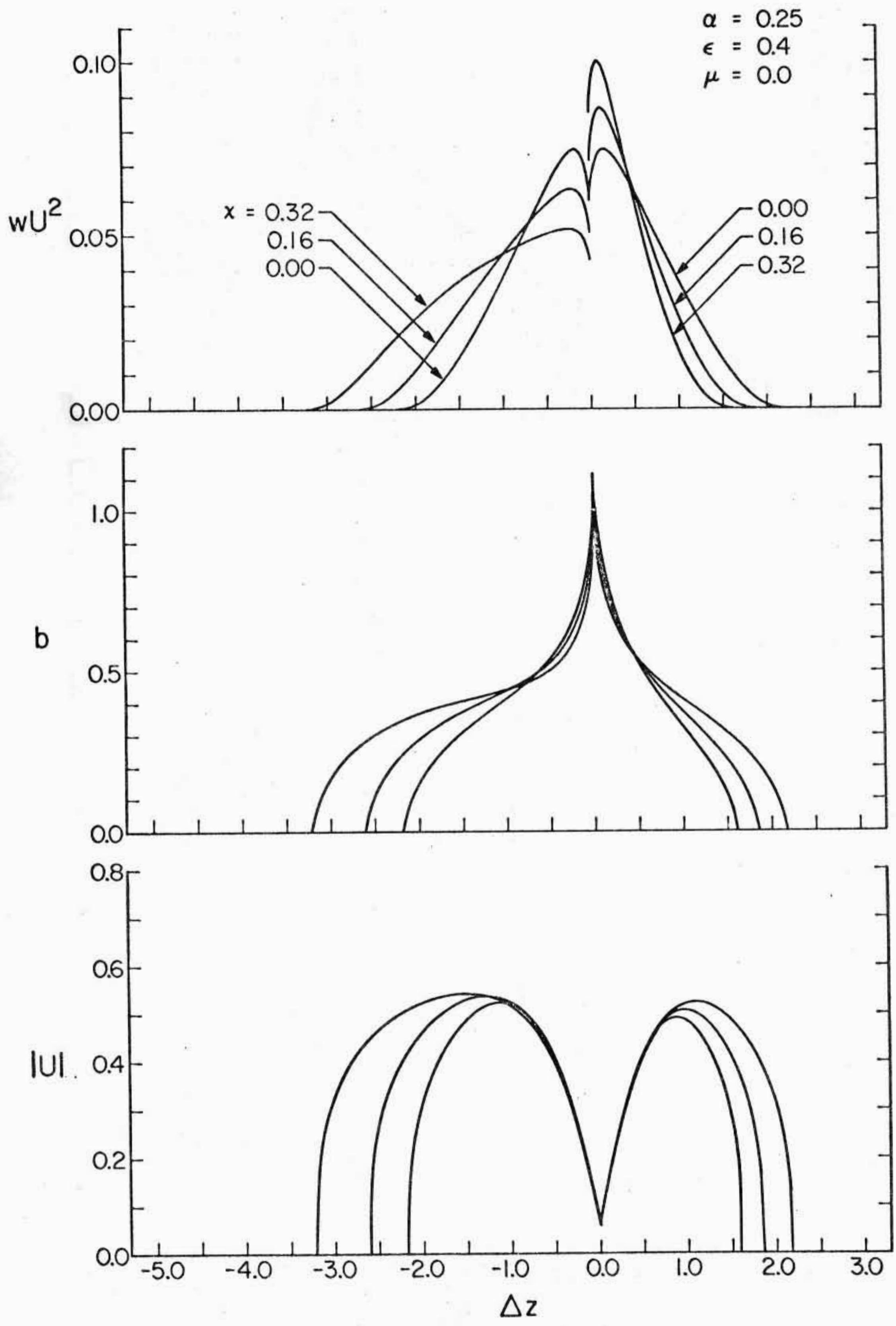

Fig. 4