MINIMAX CONTROL OF LARGE LAUNCH BOOSTERS

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Abstract

A method of specifying the gains of a linear controller for a large launch booster using a new application of optimal control theory is described in this paper. Results for a specific example are included. An important control requirement is to maintain cost variables (such as bending moment, engine gimbal deflection, and lateral deviation from desired trajectory) within specified limits in the presence of load disturbances. This requirement is met by using a performance index which depends explicitly on maximum achievable values of the cost variables in a finite time interval. The problem is formulated by using an optimal disturbance (i.e., a worst disturbance) applied to a piecewise-constant approximation of the time-varying plant. The performance of the resulting controller used with a continuous representation of the time-varying plant and subjected to typical but severe disturbances is found to be good for the following reasons:

- Gimbal deflections remain well within the specified limit;
- Lateral deviation from nominal trajectory remains well within the specified limit;
- Peak values of bending moment are close to the allowable maximum, exceeding it only under extreme conditions for which it is to be expected that improved performance could be obtained by further application of the techniques described.

I. Introduction

The problem considered is that of determining a satisfactory linear controller for a large launch booster by a new application of optimal control theory. One important requirement is to maintain a significant physical quantities (such as bending moment, gimbal deflection, and deviation from desired trajectory) within specified limits in the presence of load disturbances. This requirement is met by using a performance index (cost function) which depends explicitly on the maximum absolute values the physical quantities can achieve in a finite time interval. Harvey[1], using optimal control theory, has formulated the problem using an optimal disturbance, (i.e., a worst disturbance) and presents a method for computing the value of the cost functional for a plant with a linear controller. This theory, called "minimax theory", has been applied to the launch booster problem.

Vehicle and aerodynamic parameter variations which occur during powered flight must also be considered. These parameters are approximated by piecewise-constant functions, and results indicate that this method is satisfactory. The description of the particular launch booster studied is taken from a document from Marshall Space Flight Center entitled, "Model Vehicle No. 2 for Advanced Control Studies"[2]. In this paper, the document is referred to as the data package.

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II. Summary of Theory

The following optimal control problem is considered. It is assumed that the plant is represented by the following vector differential equation (where $\dot{x}$ denotes dx/dt):

$$\dot{x} = Ax + Bu + Cg, \quad x(0) = x^o \tag{1}$$

In this equation, x is an n-vector with components x_j describing the state of the system, x^o is the given initial condition, u is an m-vector representing the control inputs, and g is a k-vector representing the disturbance inputs. A, B, and C are constant matrices of appropriate sizes. Classes of allowable controllers and disturbances are assumed to be as follows: Ω is a class of linear fixed-gain controllers; i.e., $u \epsilon \Omega$ if:

$$u = Qx + Rg \tag{2}$$

where Q and R are constant matrices of appropriate sizes; the class of disturbances G is defined by:

$$G = \left\{ g(t): a_j \leq g_j(t) \leq b_j,\ a_j < b_j,\ j = 1, 2, \ldots, k; \quad t \epsilon [0, T],\ g(t) \text{ measurable} \right\}$$

Associated with the class of disturbances G is a constant vector h (the mean disturbance of the class) defined by $h_j = (a_j + b_j)/2$, $j = 1, 2, \ldots, k$.

To define the performance index, it is assumed that weighting n-vectors $d(i)$ and k-vectors $f(i)$, $i = 1, 2, \ldots, s$ are given, where $d(i)$ and $f(i)$ are independent of t, x^o, and g for each i. For $u \epsilon \Omega$, the cost functional $C(u)$ is defined as

$$C(u) = \max_{1 \leq i \leq s} C_i(u) \tag{3}$$

where the cost items $C_i(u)$ are:

$$C_i(u) = \max_{[0,T]} \max_{g \epsilon G} \left| d(i) \cdot x(t; x^o, u, g) + f(i) \cdot g \right|, \quad i = 1, 2, \ldots, s \tag{4}$$

with $x(t; x^o, u, g)$ representing the solution to with controller (2). An optimal controller is a ε Ω which minimizes C(u). In the computational algorithm for synthesizing an optimal controller, costs are computed for a finite set of u's $\varepsilon\ \Omega$. An optimal control for this finite set is called a minimax controller.

Substitution of (2) into (1) gives the closed-loop equation

$$\dot{x} = A_Q x + C_R g$$

where A_Q = A+BQ and C_R = C+BR. Harvey has shown that

$$\max_{t\varepsilon[O,T]} \max_{g\varepsilon G} \left| d(i) \cdot x(t; x^o, u, g) \right| = \max_{t\varepsilon[O,T]} \left\{ \left| \lambda_i(t) \right| + \mu_i(t) \right\} \quad (5)$$

where

$$\lambda_i(t) = \lambda_i(t; x^o, u) = d(i) \cdot e^{A_Q t} x^o + \int_o^t d(i) \cdot e^{A_Q \tau} C_R h \, d\tau \quad (6)$$

$$\mu_i(t) = \mu_i(t; u) = \sum_{j=1}^{k} \frac{b_j - a_j}{2} \int_o^t \left| d(i) \cdot e^{A_Q \tau} C_R(j) \right| d\tau \quad (7)$$

and $C_R(j)$ is the j-th column of C_R.

It is seen from (6) that $\lambda(t)$ is independent of individual disturbances g(t), and depends on the initial condition x^o and the mean h of all disturbances. From (7) one sees that the disturbances which maximize (5) have all bang-bang components. At the instant of time t (and corresponding g) for which (7) is maximum, one can change, if necessary, the signs of the components of g without changing the result of the maximizing process for μ_i. Therefore, the values of (4) may be computed by:

$$C_i(u) = \max_{t\varepsilon[O,T]} \left\{ \left| \lambda_i(t) \right| + \mu_i(t) \right\} + \max_g \left| f(i) \cdot g \right| \quad (8)$$

Harvey has also shown that everything necessary for the computation of $\lambda_i(t)$ and $\mu_i(t)$ can be obtained by integrating n(k + 2) + 2s first order differential equations, all but s of which are linear and the remaining s are piecewise linear. The required integration can be readily carried out on a high speed computer.

The cost functional (3) is somewhat more general than the one of the same notation in reference 1 because the term f(i) $\cdot$ g in (4) (equivalently max $[f(i) \cdot g]$ in (8)) is not considered there. This extension of the cost functional was necessary to consider gimbal angle and bending moment as cost items in the formulation of the launch booster problem presented in this paper.

The weighting vectors d(i) and f(i) will be assumed to have the following form (where superscript T denotes transpose):

$$d^T(i) = D_i \left[e_1(i) \ldots e_j(i) \ldots e_n(i) \right],$$

$$f^T(i) = D_i \left[e_{n+1}(i) \ldots e_{n+k}(i) \right] \quad (9)$$

For each i, D_i is constant. The components $e_j(i)$ may depend on controller gains, flight conditions, etc, but not on time.

The results can be extended to the case where the matrices A, B, C, Q, R have variable, rather than constant, elements. The second term of (6) becomes

$$\int_o^t d(i) \cdot \Phi(t) \Phi(\tau)^{-1} C_R \; h \, d\tau$$

and one of the integrals in (7) becomes

$$\int_o^t \left| d(i) \cdot \Phi(t) \Phi^{-1}(\tau) C_R(j) \right| d\tau$$

where $\Phi(t)$ satisfies

$$\dot{\Phi}(t) = A_Q(t) \Phi(t), \quad \Phi(0) = 0$$

Since t appears in the integrands as well as in the upper limit of the integrals, these integrals cannot be evaluated by a single integration over the interval $[O, t]$. Computing the cost items for a time-varying problem would require the evaluation of these integrals for each t ε $[O, T]$. The expense of computing cost items for a single time point t is at least as much as for the computation of cost items over the interval $[O, T]$ for a constant coefficient plant. The goal of the study was to make a step toward engineering application. Therefore, a requirement of efficiency dictated the use of the constant coefficient theory. In the evaluation in Section VII, however, a time-varying representation of the plant is employed.

III. Interpretation of Cost Functional

The variables in the cost items, from Equations (4) and (9), are seen to have the following form:

$$d(i) \cdot x + f(i) \cdot g = D_i \left(\sum_{j=1}^{n} e_j(i) x_j + \sum_{j=1}^{k} e_{n+j}(1) \, g_j \right) \quad (10)$$

The quantity in parentheses in (10) is called a cost variable y_i; and the corresponding cost item is the weighted maximum absolute value of the cost variable which can occur in the finite time interval $[O, T]$ when the plant has controller u and encounters the optimal disturbance g. Since the weighting scaler D_i is constant, it does not influence the maximizing process; and the cost items are seen to have the meaning shown in (11).

$$C_i = D_i \max_{[0,\,T]} \max_{g \in G} \left| y_i(t; x^o, u, g) \right|, \; i = 1, 2, \ldots, s. \tag{11}$$

Any physical quantity representable as a linear combination of components of the state vector and disturbance vector can be a cost variable. The cost variables selected for consideration in the control cost functional (3) will govern the selection of $e_j(i)$'s. The relative weight to be given to each cost variable is determined by the values chosen for the D_i's.

Choice of the D_i's and $e_j(i)$'s amounts to specifying the weighting vectors d(i) and f(i). This constitutes an important part of the subjective input to the minimax control design procedure. This choice is important since changing a particular weighting vector d(i) and/or f(i) in (4) will alter $C_i(u)$. This may alter the choice of the C_i which is selected in (3) as the cost C, and thus alter the choice of the "best" controller (the one which minimizes C).

The cost functionalC(u) is a good one for booster control design. For example, if

$$\left| y_i \right|_{max}, \; i = 1, 2, \ldots, s$$

denote the maximum permissible values of the cost variables, and if the weighting scalers D_i are chosen as shown in (12), then any controller u whole cost does not exceed N restrains all cost variables to within acceptable limits, even when the vehicle is subjected to the worst disturbances.

$$D_i = \frac{N}{\left| y_i \right|_{max}}, \; i = 1, 2, \ldots, s. \tag{12}$$

IV. Interpretation of Weighting Vectors

To facilitate interpretation of the weighting vectors d(i) and f(i) defined in (9), additional notation conventions will be adopted. Recall that: 1) n is the number of components in the state vector (i.e., n is the dimension of the state space), 2(each d(i) is an n-vector, 3) each f(i) is a k-vector, and 4) there is a total number s of each (i.e., i = 1, 2, ..., s). Assume that $s \geq n$ so that there are at least as many cost items C_i as the dimension of the state space. Then assume that the state variables are the first n cost variables. More complicated cost criteria will be assigned index values $i > n$.

Interpretation of d(i), f(i), i = 1, 2, ..., n

To weight state variables individually, define the first n of the weighting vectors d(i) and f(i) as follows:

$$e_i(i) = 1, \; i = 1, 2, \ldots, n$$

$$e_j(i) = 0, \; j = 1, 2, \ldots, n, \; j \neq i$$

$$e_j(i) = 0, \; j = n+1, \ldots, n+k \; (\text{i.e., } f(i) = 0).$$

Then choose D_i, i = 1, 2, ..., n according to (12).

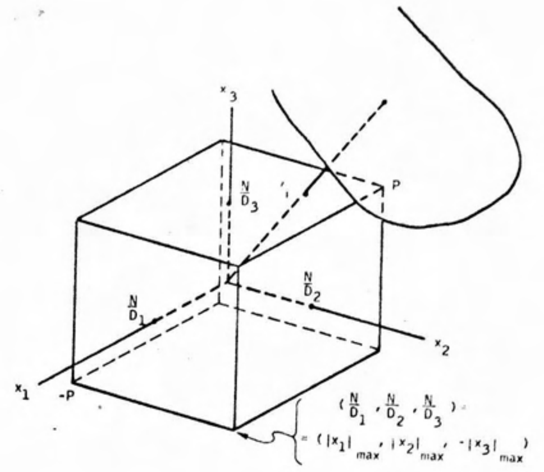

Figure 1. State Space Box of Control Cost N and Weighting Scalars D_i

Figure 1 illustrates the result on a three-dimensional state space of selecting d's and f's in this manner, but some general notation is retained. All cost items C_i, $1 \leq i \leq n$ for points $x = (x_1, \ldots, x_i, \ldots, x_n)$ inside the box are less than the specified cost N. Thus, a controller which keeps a response trajectory in the box is one whose cost C does not exceed the value N. Increasing the value of D_i reduces the maximum amplitude of the state variable x_i that can be allowed by a controller whose cost C is not to exceed N. Alternately, increasing D_i increases the cost item C_i associated with state variable x_i for a given set of controller gains. Thus a larger D_i gives a larger weight to x_i in determining control cost C.

If no restriction is to be placed on x_i in determining an acceptable controller, simply let $D_i = 0$. The effect is to stretch the state space box to infinite length in the x_i direction.

Interpretation of d(i), f(i), $n < i \leq s$

Control cost elements C_i, $n < i \leq s$, which are linear combinations of two or more state variables will restrict the state space for a given cost N, if D_i is large enough.

For example, consider the cost item C_i, $i > n$, which is associated with a control law u given by (13).

$$u = \sum_{j=1}^{n} K_j x_j \tag{13}$$

The $e_j(i)$'s are specified according to the rule

$$e_j(i) = K_j, \quad j = 1, 2, \ldots, n$$

$$e_j(i) = 0, \quad j = n+1, \ldots, n+k \; (\text{i.e., } f(i) = 0) \tag{14}$$

As with state variables, it is common to specify a maximum value $|u|_{max}$ for the control (for example, $|u|_{max}$ may be allowable gimbal deflection). D_i for this cost variable is chosen by using $|u|_{max}$ in the denominator of (12).

It remains to see what this cost item does in the way of restricting the state space box. The control

$$u = \sum_{j=1}^{n} e_j(i)\,x_j$$

defines a family of parallel planes(hyperplanes if $n > 3$) in the state space. The two planes which represent maximal control are given by

$$\sum_{j=1}^{n} e_j(i)\,x_j \pm \frac{N}{D_i} = 0 \qquad (15)$$

The two planes of the same family which pass through corners P and -P of the state space box N (Figure 1) are represented by

$$\sum_{j=1}^{n} e_j(i)\,x_j \pm \ell_i = 0 \qquad (16)$$

where ℓ_i is the distance from the origin to these planes.

It is a matter of geometry to show that ℓ_i may be computed as follows:

$$\ell_i = \sum_{j=1}^{n} |e_j(i)|\ |x_j|_{max} / \left(\sum_{j=1}^{n} e_j^2(i) \right)^{1/2} \qquad (17)$$

Comparing (15) and (16), one sees that the control planes of cost N pass through corners of the state space box of cost N if $\ell = N/D_i$. Therefore, if $D_i > N/\ell_i$, the maximum control planes cut corners off the box and further restrict the state space which was established by allowing cost N or less on the individual state variables.

Cost Items Depending on State Variables and Disturbances $n < i \leq s$

It is occasionally necessary to consider disturbances in control cost items also. In such cases, the f(i)'s are no longer zero. The additional cost due to the disturbance entering a cost variable is given in the last term of (8); that term is $\max_g |f(i)\cdot g|$. This additional cost should simply be omitted in considering the effects of a particular d(i) on the state space box of cost N (the vector f(i) does not effect it). The preceding results in this section then apply directly.

V. Launch Booster Equations

Equations of Motion

An important control criterion for larch launch boosters is to minimize structural bending. The major causes of bending are engine gimbal deflection and attack angle. Most of the bending moment due to these causes is associated with the rigid body part of the equations of motion. So a rigid body approximation of Model Vehicle No. 2 was used, and bending moment, as a cost variable, was weighted heavily in the cost functional.

A set of second order linear differential equations of longitudinal motion with open loop control was taken from the data package. Table 1 describes the notation of the data package. In the notation of the "Summary of Theory" section of this paper, the state vector x, control u, and disturbance $g(t, a_i, b_i)$ are defined as:

$$x^T = [\,\phi\ \dot{\phi}\ \dot{Z}\,]\ ,\ u = \beta$$

$$g = V_w : a_1 = -|V_w|_{max},\ b_1 = |V_w|_{max}$$

$$a_j = b_j = 0,\ j \neq 1;\ h_j = 0,\ \text{all } j \qquad (18)$$

The controller class is that of linear fixed-gain controllers of the form:

$$\beta = K_1\,\phi + K_2\,\dot{\phi} + (K_3/V)\,\dot{Z} - (K_3/V)\,V_w \qquad (19)$$

It is seen by comparing (18), (19), and (2) that

$$Q = [K_1\ K_2\ K_3/V] \text{ and } R = -K_3/V.$$

It is important to note that since attack angle α is given by

$$\alpha = \phi - \dot{Z}/V + V_w/V = \phi - \dot{Z}/V + \alpha_w, \qquad (20)$$

(19) is equivalent to a controller using pitch attitude ϕ, pitch rate $\dot{\phi}$, and attack angle α as shown in (19A).

$$\beta = (K_1 + K_3)\,\phi + K_2\dot{\phi} - K_3\alpha \qquad (19A)$$

Furthermore, by a more complicated manipulation, (19) can be shown to be equivalent to a linear controller with pitch attitude ϕ, pitch rate $\dot{\phi}$, and normal acceleration A_a information, where the normal accelerometer is located at x_A. Thus, it is seen that the form of the matrix R is determined by the instrumentation chosen to supply data to the controller. Similarly, the form of Q is determined by the equation chosen for the controller itself. A given controller is determined by assigning compatible values to the elements of Q and R. Note that it is also assumed that the gimbal motor perfectly reproduces the control signal.

Table 2 contains the details of the notation for matrices A, B, and C of (1). The resulting closed-loop equation $x = A_Q x + C_R V_w$ (where $A_Q = A + BQ$ and $C_R = C + BR$) is given as follows:

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \ddot{Z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ a_{21} + b_2K_1 & b_2K_2 & (a_{23} + b_2K_3)\frac{1}{V} \\ a_{31} + b_3K_1 & b_3K_2 & (a_{33} + b_3K_3)\frac{1}{V} \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \\ \dot{Z} \end{bmatrix} + \begin{bmatrix} 0 \\ (c_2 - b_2K_3)\frac{1}{V} \\ (c_3 - b_3K_3)\frac{1}{V} \end{bmatrix} V_w \qquad (21)$$

Values of Coefficients and Disturbances

Since minimax theory in its present state may be applied efficiently only to constant coefficient plants, the approach reported here has been to divide the first 84 seconds of flight into seven intervals (flight conditions) and to approximate the time-varying vehicle by a different set of constant coefficient differential equations in each of the seven intervals.[3] The fifth interval contains the event of Mach 1 and the seventh that of maximum dynamic pressure. Inspection of Table 2 shows that the following coefficients and the values of velocity are sufficient to describe all time-variable elements of (21):

$$a_{21},\ a_{31},\ a_{33},\ b_2,\ b_3.$$

The most striking variation with time is observed by plotting $\frac{a_{21}}{V}$. This is shown in Figure 2 as a broken-line graph connecting the points eight seconds apart for which all information was available from the data package. The horizontal line segments show the seven intervals into which the first 84 seconds of flight were subdivided, and the values of the piecewise constant approximation of $\frac{a_{21}}{V}$ over the interval spanned by each segment. Velocity and the remaining coefficients all were monotone for at least the first 80 seconds. They also were approximated by step functions over the same intervals shown in Figure 2.

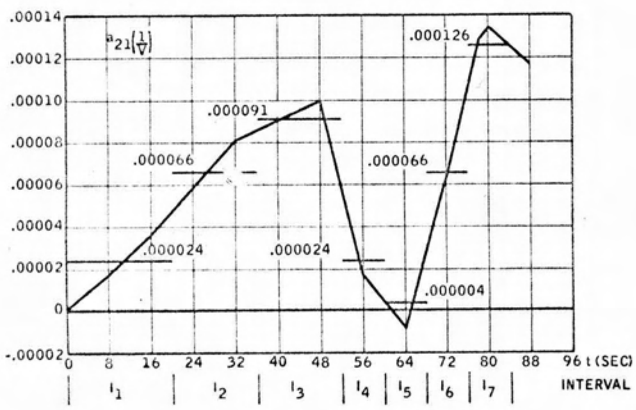

Figure 2. One Coefficient versus Time

For the disturbances, values of wind velocity magnitudes were taken from a "ninety-five percent probability of occurrence wind speed profile envelope" in the data package. The assumed winds had a mean velocity of zero and amplitudes which represent an attack angle due to wind of about 10 degrees in most intervals. These values, combined with the fact that they were bang-bang disturbances, constitute a severe disturbance class.

Bending Moment

Bending moment on the structure was weighted heavily as a control criterion in this study. It was computed as in (22). The values of the

$$M_B = (M'_\alpha)\,\alpha + (M'_\beta)\,\beta \qquad (22)$$

coefficients M'_α and M'_β depend on longitudinal location on the vehicle and other factors which vary with time. Graphs of M'_α and M'_β at t = 72 and t = 78 (instant of max q) were used, together with elementary theory, to estimate peak values of M'_α and M'_β at any instant in the interval [O, 84]. Thus, an estimate of peak bending moment on the structure at each instant of time was used in formulating control criteria. It is of interest to note that supplementary data on peak values of M'_α and M'_β which was received from Marshall Space Flight Center too late to be used in the computations shows the assumptions to be substantially correct. Bending moment is expressed in terms of state variables and disturbance winds V_w by substitugint (19) and (20) into (22) for β and α respectively. The result is (23)

$$M_B = \left[(M'_\alpha + K_1 M'_\beta) \quad K_2 M'_\beta \quad \frac{(-M'_\alpha + K_3 M'_\beta)}{V} \right] \begin{bmatrix} \phi \\ \dot{\phi} \\ \dot{Z} \end{bmatrix} + \frac{(M'_\alpha - K_3 M'_\beta)}{V} V_w \qquad (23)$$

VI. Minimax Cost Computations

Computing Procedure

The minimax computing procedure[3] consists primarily of an iterative technique. Control costs are evaluated for a selected set of controllers. The cost information is then used to select a new set of controllers, some of which have costs less than any in the preceding set. Several iterations of this procedure have been found useful in that each iteration has yielded controllers whose costs were lower than any previously considered controller.

The primary goal was to study the applicability of minimax theory to selection of controllers; consequently, no restrictions (such as asymptotic stability) were placed on the class of controllers to be considered, and no attempt was made to devise automatic numerical techniques of determining lowest cost controllers, although the procedure described here provides motivation for making the iterations automatic.

A geometric interpretation is useful for selecting a new set of controllers at each stage of cost iteration. The control class U is defined by (19) and a particular controller is therefore specified by values of the gains K_1, K_2, K_3. The following discussion is directly applicable to this situation, but n-dimensional notation is employed with a geometric example shown in three dimensions.

It will be assumed that the control criteria and weighting vectors have been chosen so that cost elements $C_i(u)$ of (4) are defined. The gain space, which will also be called U, is the n-dimensional space whose points K are the n-tuples of numbers K_j, j = 1, 2, . . . , n which comprise one combination of controller gains:

$$(K_1, K_2, \ldots, K_n) = K \in U.$$

A controller is assumed to be completely specified by selecting a point K. Thus, it is not ambiguous to speak interchangeably of a controller u = u(K) and a point K in gain space U.

The following steps constitute the basic minimax computing procedure which was used:

1. Choose a first set of gains $K^{(1)} = (K_1^{(1)}, \ldots, K_n^{(1)})$ by any reasonable means.
2. Pick a box in gain space of generous size by selecting a couple of values of values of each gain on both sides of those in $K^{(1)}$. Use all combinations of the selected values to define a grid of points uniformly distributed in the box. Figure 3 illustrates such a "box of controllers".

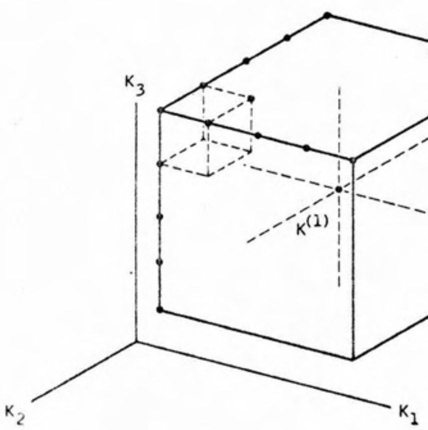

Figure 3. Box of Controllers in Gain Space

3. Compute the cost of control for the points of this first grid and see where a few of the lowest cost controllers fit in the grid.
4. If any particular gain K_j is inside the box for the low cost controllers, this is an indication of a somewhat critical range of values for that gain, and a second grid with a refinement and reduced range of values in the j-th direction in gain space should be selected and the computations repeated.
5. If any particular gain K_j is on the surface of the box for the low cost controller, this is an indication that the next grid should have its range of values shifted the j-th direction.

Steps 2 through 5 are iterated until there is little cost reduction between iterations. At any state of iteration, the minimax controller is the one of those considered with the lowest control cost.

Weighting Vectors and Scalars Selected

The state variables ϕ, $\dot{\phi}$, and $\dot{Z}$, together with gimbal deflection β and bending moment M_B were chosen as cost variables for the problem outlined in Section V. Table 3 shows the definitions of $e_j(i)$'s and D_i's.

Data was supplied by Marshall Space Flight Center which included closed-loop responses of Model Vehicle No. 2 to wind disturbances when equipped with a drift-minimum controller.[4] These particular disturbances, called "synthetic wind profiles", are described in Section VII. The values of the weighting scalars D_i were selected for each flight interval shown in Figure 2 so that maximum responses of the vehicle for wind profiles appropriate to that interval all had the same cost. Then the resulting bending moment weights D_5 were doubled, while the others were left unchanged. The bottom half of Table 3 shows the D_i's for each flight interval.

The method of choosing the D_i's accomplished two objectives:

- The method distributed the weights with recognition of physically realistic values, and
- The method enabled the resulting cost items to be compared directly and interpreted against a known allowable number

  (e.g., if $D_i = \dfrac{1}{|y_i|_{max}}$,

  then $D_i \cdot |y_i| \le 1$).

Minimax Controllers

Minimax controllers were first determined for each of the seven constant coefficient representations of the vehicle, one for each interval shown in Figure 2. The initial condition of $x^o = 0$ was used in each case.[3] In this part of the study, it was found that the controller in each interval gave a closed loop plant with three real poles of which one was positive. This positive pole had very small magnitude in all intervals except the fifth, which includes the event of Mach 1. The effect of non-zero initial conditions on the selection of minimax controllers was then determined for third and fifth intervals, using the $x^o = 0$- controllers as the starting point in gain space.[5] The resulting minimax controllers for these two intervals were "universal" controllers in that they had low cost for a typical range of initial conditions. The significance of this can be realized by first noting that each initial condition will have its own minimax controller. However, it was found that increasing the magnitude of a given initial condition almost always caused a monotone shift in the individual gains to the new minimax controller; and that the shift in the direction of the gains was almost always the same for all initial conditions. Thus, controllers which had low cost for a reasonable range of initial conditions were found.

The resulting minimax controller for the third interval gave a plant with closed loop poles consisting of a complex pair and a real pole which was very slightly negative. The new fifth-interval controller still had all real poles; one was still positive, but its magnitude was substantially less than the controller determined for zero initial conditions.

The minimax controllers determined with non-zero initial conditions were stable (or less unstable) than those for $x^o = O$, and the change in pole location was more pronounced in the fifth interval (which contains the event of Mach 1) than in the subsonic flight condition of the third interval. The shift toward stability is to be expected. Equation (6) shows that cost of a given controller will increase if the initial condition x^o goes from zero to a non-zero value. In the face of non-zero initial conditions, a new controller which is stable (or less unstable) can be expected to have lower total cost. This is because the part of its cost depending on initial conditions will be smaller than for the controller which is optimal if $x^o = O$.

Table 4 shows the gains, the cost items for $x^o = O$, and the closed loop poles of the seven minimax controllers (the universal controllers were used in intervals three and five, but their costs for $x^o = O$ are listed). Table 4 also contains the cost items for drift minimum controllers encountering optimal disturbances. These drift minimum controllers have one pole at the origin (to satisfy the drift minimum condition) and a complex pair with a damping ratio of 0.7 and an undamped natural frequency of 0.2 cycles per second. Since Model Vehicle No. 2 was synthesized for study of the bending problem, the importance of low values of the bending moment cost item C_5 is self-evident. In table 4, the values of C_5 for the minimax controllers are much less than for the drift minimum controllers. Here is one indication for preference of the minimax criterion over more conventional criteria.

VII. Responses to Typical Winds

The minimax computing procedure leads to controller parameter values which minimize vehicle response to optimal (worst) disturbances. For another measure of the applicability of this procedure, responses to typical disturbances of the vehicle with a minimax controller were computed.[6]

Figure 4 below shows the piecewise constant gain functions obtained by juxtaposing the minimax controller gains listed in Table 4 for each interval. A continuous-gain controller (shown by broken lines in Figure 4) is synthesized by linear interpolation over each four-second interval containing a discontinuity in gain. This continuous-gain controller is used with a continuous-coefficient representation of the plant, and responses to five typical wind disturbances are computed.

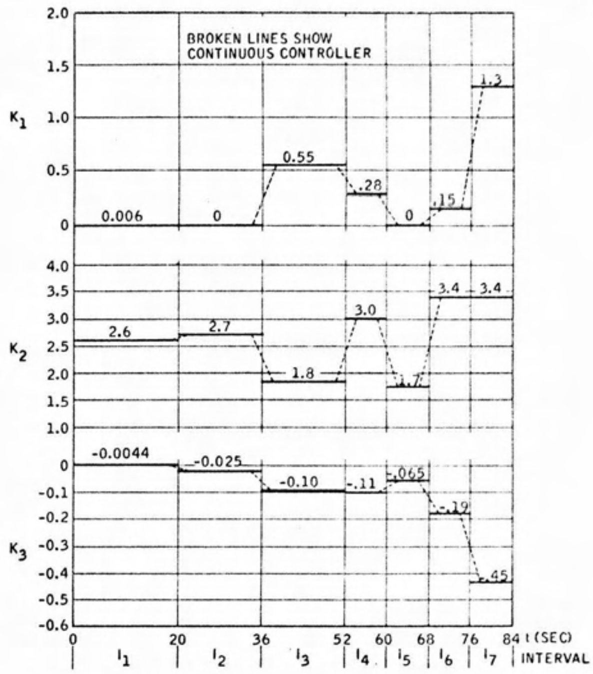

Figure 4. Minimax Controller Gains

Synthetic Wind Profiles

Five synthetic wind profiles[4] are used as load disturbances for computing time responses of the closed-loop plant. Each profile is specified in terms of attack angle due to wind α_W, and is formed by superposing a gust on a basic profile. Each basic profile is identified by the time T after launch when α_W reaches its maximum value, slightly exceeding 10 degrees. (The amplitude, in terms of α_W, of the optimal disturbances used in the minimax calculations also slightly exceeded 10 degrees in the first five intervals. It was about 9 and 8 degrees in intervals 6 and 7 respectively.) Maximum wind build-up rates, which depend on altitude, govern the increase of α_W from its initial value of zero to its value at time T. The value of $\alpha_W(T)$ persists for an altitude layer of 3 kilometers, and then decreases as fast as it increased previous to time T. The five basic profiles reach their maximum values at T = 48, 56, 64, 72 and 80. On each one except the first, a gust with an altitude depth of 0.3 kilometers is superposed at time T. The gust has very short rise and drop-out times, and the α_W describing the composite wind profile is still a continuous function of time. The amplitude of the gust depends on altitude. Figure 5 is a graph of wind profile $\alpha_W(56)$, with labels showing the qualitative properties of all the profiles.

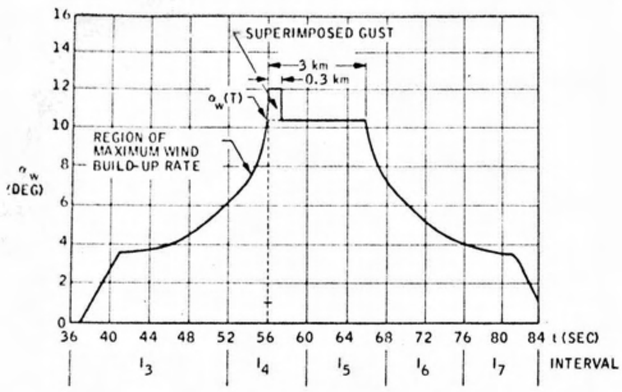

Figure 5. Synthetic Wind Profile $\alpha_W(56)$

The synthetic wind profiles represent severe wind disturbances, but are good approximations of distrubances which can occur in nature. Since the optimal disturbances cannot occur in nature, the synthetic wind profiles are referred to as "typical".

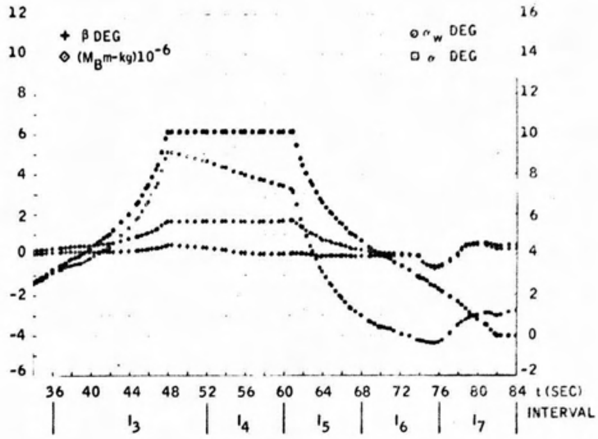

Figure 6. Time Responses for Continuous Plant, Continuous Gains, Continuous Wind α_W (48)

Effect of Initial Conditions

Near the end of the Section entitled "Minimax Cost Computations", it was seen that the controller which is minimax for zero initial conditions will usually not be the best one if $x^0 \neq 0$. In computing time solutions with typical disturbances, the initial conditions will be zero only in the intervals up to and including the one where the disturbance begins. Some supplementary work[6] which cannot be reported here for lack of space shows that the effect of initial conditions on response amplitudes is small in the first three or four intervals. In the sixth and seventh intervals (supersonic flight conditions at high dynamic pressure) initial conditions have a greater effect, and should have been considered in the minimax design calculations. In the fifth interval, where they were considered, it proved to be necessary.

Individual Response Magnitudes

Figures 6 and 7 show the responses to the $\alpha_W(48)$ disturbance. The responses to the $\alpha_W(56)$ disturbance are shown in Figures 8 and 9. Graphs for the other three responses are omitted because they reveal no additional qualitative information.

A digression of interest concerns the stability of the minimax controller. The time solutions certainly do not possess the divergent appearance that might be expected. This is explained by the fact that the coefficient in the time solution of the positive exponential function (corresponding to the small positives pole) is sufficiently small that this part of the solution does not grow much in the finite time interval over which the controller operates. It illustrates that asymptotic stability is not an absolute requirement of a controller whose time interval of operation is short. It is reasonable to expect something in return for sacrificing asymptotic stability. In this case, the "return" is a

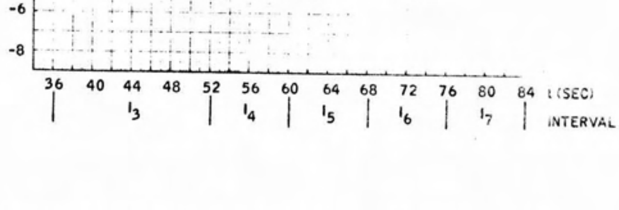

Figure 7. Time Responses for Continuous Plant, Continuous Gains, Continuous Wind α_W (48)

controller which keeps bending moments acceptably small without acquiring large trajectory errors. These facts are demonstrated in the next few paragraphs.

The amplitudes of the cost variables in intervals three through seven for all five wind profiles are shown in Table 5. (Response amplitudes in the first and second intervals were zero or too small to be of interest.) Values of the trajectory deviation Z, which was not a cost variable in the minimax studies, have been included in Table 5.

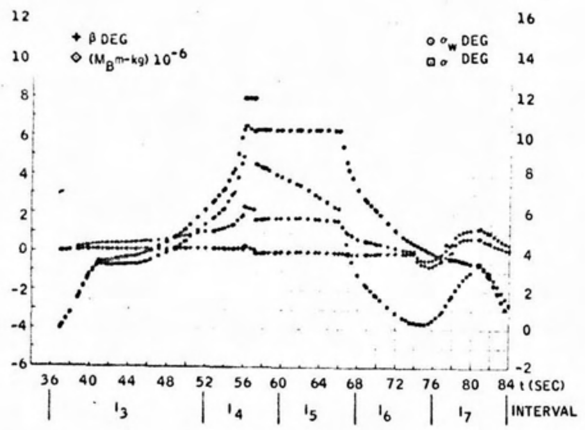

Figure 8. Time Responses for Continuous Plant, Continuous Gains, Continuous Wind α_w (56)

Quantitative performance is best judged by the amplitudes of β, M_B, Z, and, to some extent, $\dot{Z}$. Values of β must not exceed the gimbal deflection limit of 5 degrees. This corresponds to a value in Table 5 of (β rad.) x 10 = 0.873; and it is seen that this number is not approached in any interval for any of the wind profiles (fourth row in each box of Table 5). It is a reasonable conjecture that gimbal deflections would have been greater had not gimbal dynamics been neglected, but they could have been three times greater without reaching the gimbal deflection limits. Thus, the minimax controller keeps gimbal deflection well within the required limit.

A design goal is that maximum value of bending moment should not exceed 2.7×10^6 m-kg. This corresponds to a table value of (M_B m-kg) x 10^{-6} = 2.7. The table shows that this was exceeded in only the sixth interval with the $\alpha_w(72)$ disturbance. The excess was 17 percent. (A negligible excess occurred in interval seven). Since the controller gains for the sixth and seventh intervals were based on zero initial conditions, it is to be expected that improved gains would be obtained by considering non-zero initial conditions. Therefore, the maximum bending moment values for all five profiles are seen to be quite suitable.

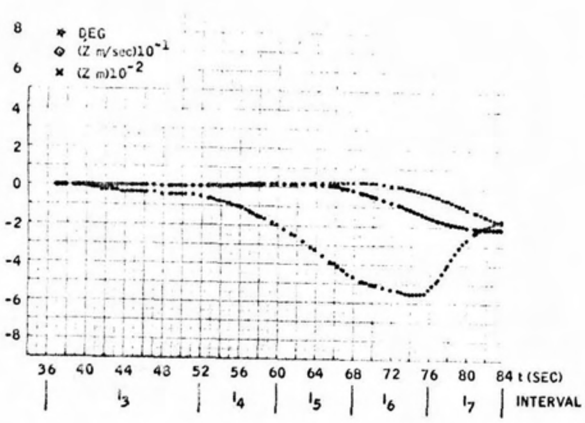

Figure 9. Time Responses for Continuous Plant, Continuous Gains, Continuous Wind α_w (56)

Table 5 also shows that the peak bending moment for each disturbance except $\alpha_w(48)$ occurred in the interval where the wind profile reached its maximum value. In fact, the bending moment peak occurred when the superimposed gust reached its peak value. The maximum bending moment for the $\alpha_w(48)$ disturbance is 3 percent higher than its value at T = 48 and occurs at the end of the peak value of wind velocity. It will be recalled that no gust was superposed on the basic $\alpha_w(48)$ profile. Note also that the bending moment peak is very distinct with the other four profiles, and substantially higher than with the $\alpha_w(48)$ disturbance. These facts indicate that the gusts are responsible for the sharp peaks in bending moment. It is also noted that the peak value of benging moment is close to the allowable maximum whenever the disturbance includes a gust. The fact that values of bending moment are small early and late in the history of the individual wind profiles is to be expected. A small bending moment should accompany a small disturbance amplitude.

It has been seen that the minimax controller considered allows bending moments which are close to the design maximum for the typical disturbances including a superposed gust (which is much like a bang-bang disturbance). The peak bending moment occurs when the disturbance has its maximum amplitude. These are strong inditions that the class of optimal disturbances do lead to controllers which give bood bending-moment performance when the vehicle is subjected to typical disturbances.

It is important to keep lateral displacement Z low for the sake of fuel efficiency and guidance accuracy. A displacement of 3000 meters at t = 150 seconds should not be exceeded. Since dynamic pressure falls off after t = 78 about as fast as it grew before that time, it is clear that wind loads will decrease substantially in the last half of powered flight. It follows that large trajectory corrections can be made without exceeding bending moment limits after, say, t = 120. Therefore, it is probably reasonable to budget a large part of the 3000 meter allowable error to the first half of the powered flight interval. But this does not even appear to be necessary. The maximum values of Z acquired by the vehicle are shown in the last row of each division of Table 5. For all profiles except $\alpha_W(72)$, the maximum Z is seen to increase with the interval. The largest value is only 200 meters. It occurs with the wind which started to blow earliest in the flight ($\alpha_W(48)$). Thus, the minimax controller is seen to keep trajectory deviations very small, even in the portion of the flight when load minimization is the most important criterion.

Low levels of lateral velocity $\dot{Z}$ are desirable for at least two reasons. One is that $\dot{Z}$ contributes to attack angle,hence to bending moment, as shown in Equations (20) and (23). The other reason is that keeping $\dot{Z}$ small is a direct way to keep small the lateral deviation Z. Thus, there is no direct criterion on the magnitude of $\dot{Z}$. The bending moment performance for the minimax controller has been seen to be suitable; and the lateral displacement performance is very good. There remains only consideration of the general $\dot{Z}$ performance. For any of the profiles, the largest values of $\dot{Z}$ occur in the later intervals when the wind velocity and dynamic pressure are high. Large magnitudes of $\dot{Z}$ occurring in intervals I_6 and I_7 for the three profiles which began earliest in time can be charged to the fact that controller gains for these intervals were not well suited to the large initial conditions encountered.

VIII. Restrictions on the Class of Allowable Controllers

One advantage of the minimax theory is that the cost computations do not require a total commitment to the use of optimal control theory. The controller can be partially specified by classical control criteria. Then iterations of cost computation can be used to optimize the remaining controller parameters.

For example, the class of allowable controllers considered in this report is given by (2), and for the specific example by (19). It is a class of linear controllers with fixed gains over finite time intervals. It might be desired that further restrictions be placed on the class from which controllers are to be selected. In Section VII, Table 4, two extremes were observed. The minimax controllers were selected solely on the basis of minimizing control cost in each (finite) flight interval. The resulting controllers (except for the third interval) were asymptotically unstable.

The drift minimum controllers were totally specified by the conditions that the controllers be drift minimum and have an undamped natural frequency of 0.2 cycles per second with a damping ratio of 0.7. Minimax computations showed these controllers to have substantially higher costs than the minimax controllers in each flight interval.

It is practical to impose conditions on the class of controllers which are intermediate between the two extremes of unrestricted cost minimization and total specification; however, imposing such restrictions will probably increase control costs. Examples of increasing degrees of restriction for controllers of the launch booster defined in Section V will serve as illustrations.

Drift-Minimum Principle Imposed

The characteristic equation of the closed loop plant will be written as:

$$\lambda^3 + A_1\lambda^2 + A_2 + A_3 = 0 \tag{24}$$

where, in the notation of Tables 1 and 2,

$$A_1 = -\left[\left(a_{33} + b_3K_3\right)\frac{1}{V} + b_2K_2\right] \tag{25}$$

$$A_2 = \left[\left(a_{33}b_2 - a_{23}b_3\right)\frac{1}{V}K_2 - \left(a_{21} + b_2K_1\right)\right] \tag{26}$$

$$A_3 = \frac{1}{V}\left[\left(a_{33}b_2 - a_{23}b_3\right)K_1 + \left(a_{21}b_3 - a_{31}b_2\right)K_3 + \left(a_{21}a_{33} - a_{23}a_{31}\right)\right] \tag{27}$$

The drift minimum principle is defined by requiring $\dot{Z} = 0$ when ϕ is quasi-steady state. It can be shown that this is equivalent to $A_3 = 0$, and it is clear from (24) that one closed-loop pole is then at the origin ($\lambda = 0$). Thus, setting (27) equal to zero gives a linear equation in K_1 and K_3 which must be satisfied to give a drift-minimum controller. Two gains remain unspecified, and iterations of minimax calculations (subject to $A_3 = 0$) could be used to fine a minimax drift-minimum controller.

A further restriction on the class of controllers defined by (19) could be to impose the drift-minimum principle and require that the two unspecified pole locations be in the left half-plane. Then one would consider minimax controllers which satisfy $A_3 = 0$ and the conditions specified in (28).

$$A_1 > 0$$
$$A_2 > 0 \tag{28}$$

It is seen by referring to (25) and (26) that (28) gives two linear inequalities which must be satisfied by the controller gains.

Asymptotic Stability Imposed

A still more restricted subclass of (19) would be controllers which are asymptotically stable. It can be shown that conditions (28) and (29) are necessary and that these, together with (30) are sufficient for asymptotic stability.

$$A_3 > 0 \tag{29}$$

$$A_1 A_2 > A_3 \tag{30}$$

Note that such a controller can still be arbitrarily close to being drift-minimum by having A_3 be sufficiently small. Minimax calculations would still be applicable in arriving at stable-minimax controllers.

IX. Conclusions

The performance of the minimax controller with typical disturbance inputs is judged to be good for the following reasons:

- Gimbal deflections remain well within the specified limits;
- Lateral deviation from the nominal trajectory remains well within the specified limit;
- Peak values of bending moment are close to the allowable maximum, exceeding it only when encountering severe gusts at high dynamic pressure with controller gains based only on zero initial conditions.

These conclusions deserve elaboration, however. Since the gimbal deflection and lateral deviation can be allowed to be substantially higher than they are, it would be preferable to assign lower weights to β and $\dot{Z}$ in the minimax design calculations while simultaneously increasing the weight on M_B. The minimax controller corresponding to this new weighting on the cost variables could be expected to give smaller values of M_B, larger values of β and Z, but still keep all maxima within the specified limits when the vehicle is subjected to the typical disturbances.

An advantage of this particular use of optimal control theory is that it does not require a total commitment to optimization. If some classical control criterion (e.g., asymptotic stability) is an overriding consideration, it can be imposed on the sets of controllers selected for cost computations. Then the controls can be optimized subject to this restriction.

The over-all conclusion is that the minimax computing procedures for a piecewise constant approximation of the plant show definite promise as a design technique. These procedures will be especially applicable in situations where a maximum response of any of the large variety of cost variables which can be considered will lead to catastrophic results. The launch booster problem is an example of just such a case. Gimbal deflection, bending moment, or laterial deviation limits must not be exceeded.

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Table 1. Table of Physical Quantities

Symbol	Description	Dimension
X	Drag Force	kg
M	Mass	$kg\text{-}sec^2/m$
q	Dynamic Pressure	kg/m^2
V	Vehicle Velocity	m/sec
F	Total Thrust	kg
I_{xx}	Moment of Inertia	$kg\text{-}sec^2\text{-}m$
x_β	Engine Gimbal Point	m
D	Diameter of Vehicle	m
C_{Z_α}	Normal Force Coefficient	
C_{D_o}	Drag Coefficient	
CP	Center of Pressure	m
CG	Center of Gravity	m
$x_{cp} = (CP + x_\beta)$	Station of CP	m
$x_{cg} = (CG + x_\beta)$	Station of CG	m
x_A	Station of Accelerometer	m
A_a	Normal Acceleration	
ϕ	Pitch Attitude Deviation from Reference Trajectory	rad
α	Attack Angle	rad
α_w	Attack Angle due to Wind	rad
Z	Displacement Perpendicular to Reference Trajectory	m
M_B	Bending Moment	m-kg
M'_α	Bending Moment Coefficient	m-kg/rad
M'_β	Bending Moment Coefficient	m-kg/rad
V_W	Speed of Wind Perpendicular to Reference Trajectory	m
β	Gimbal Deflection Angle	rad

Table 2. Elements of Matrices A, B, C in Terms of Vehicle and Trajectory Parameters

$a = -\dfrac{C_{Z_\alpha}}{C_{D_O}} X$	$a_{33} = \dfrac{a}{M}$
$a_{21} = \dfrac{a(x_{cg} - x_{cp})}{I_{xx}}$	$b_2 = -\dfrac{F}{2} \dfrac{(x_{cg} - x_\beta)}{I_{xx}}$
$a_{31} = (F - X - a)\dfrac{1}{M}$	$b_3 = \dfrac{F}{2}\dfrac{1}{M}$
$a_{12} = 1$	$c_2 = -a_{23} = a_{21}$
$a_{23} = -a_{21}$	$c_3 = -a_{33}$
$a_{11} = a_{13} = a_{22} = a_{32} = b_1 = c_1 = 0$	

Table 3. Cost Items and Weight Vectors

i	1	2	3	4	5
y_i	ϕ rad	$\dot{\phi}$ rad/sec	$\dot{Z}$ m/sec	β rad	M_B m-kg
$e_1(i)$	1	0	0	K_1	$M'_\alpha + K_1 M'_\beta$
$e_2(i)$	0	1	0	K_2	$K_2 M'_\beta$
$e_3(i)$	0	0	1	K_3/V	$(-M'_\alpha + K_3 M'_\beta)/V$
$e_4(i)$	0	0	0	$-K_3/V$	$(M'_\alpha - K_3 M'_\beta)/V$
Scalar	D_1	D_2	D_3	D_4	D_5
I_1	0.978	0.01	0.0342	1.56	0.674×10^{-7}
I_2	0.978	0.01	0.0342	1.56	0.398×10^{-7}
I_3	0.645	0.01	0.0262	0.838	0.166×10^{-7}
I_4	0.529	0.01	0.0188	0.684	0.120×10^{-7}
I_5	0.499	0.01	0.00859	0.763	0.104×10^{-7}
I_6	0.415	0.01	0.0143	0.608	0.0928×10^{-7}
I_7	0.493	0.01	0.0314	0.469	0.1072×10^{-7}

Table 4. Gains and Closed-Loop Poles of Minimax Controllers and Cost Items of Minimax and Drift Minimum Controllers

t		Minimax Gains		Minimax Poles	i	Cost Items Minimax C_i	Cost Items Drift Minimum C_i
0	I_1	K_1	0.006	0.00640	1	0.00119	0.00251
					2	0.0000007	0.00003
		K_2	2.6	-0.00964	3	0.01566	0.00243
					4	0.00204	0.04013
		K_3	-0.0044	-0.90913	5	C = 0.01624	C = 0.07383
20	I_2	K_1	0	0.02264	1	0.00967	0.0096
					2	0.000007	0.0001
		K_2	2.7	-0.02414	3	0.03580	0.0104
					4	0.01125	0.1518
		K_3	-0.025	-0.99438	5	C = 0.03608	C = 0.1666
36	I_3	K_1	0.55	-0.00018	1	0.01371	0.0152
					2	0.00008	0.0003
		K_2	1.8	-0.3587	3	0.03747	0.0214
				±i0.2813	4	0.02855	C = 0.1835
		K_3	-0.10		5	C = 0.04523	0.1619
52	I_4	K_1	0.28	0.01268	1	0.01609	0.0172
					2	0.00007	0.0004
		K_2	3.0	-0.12597	3	0.04673	0.0238
					4	0.02615	C = 0.1870
		K_3	-0.11	-1.1651	5	C = 0.04737	0.1576
60	I_5	K_1	0	0.04179	1	0.02093	0.0228
					2	0.00006	0.0005
		K_2	1.7	-0.05921	3	0.03493	0.0153
					4	0.01778	C = 0.2931
		K_3	-0.065	-0.7141	5	C = 0.04441	0.1928
68	I_6	K_1	0.15	0.02659	1	0.01587	0.0156
					2	0.00006	0.0004
		K_2	3.4	-0.06544	3	0.04959	0.0235
					4	0.03214	C = 0.1800
		K_3	-0.19	-1.5435	5	C = 0.05017	0.1416
76	I_7	K_1	1.3	0.00076	1	0.01686	0.0182
					2	0.00018	0.0004
		K_2	3.4	-0.51510	3	0.07417	0.0517
					4	0.05525	0.1443
		K_3	-0.45	-1.0713	5	C = 0.07525	C = 0.1662
84							

Notation:

$C_i = D_i |y_i|$ $y_1 = \phi$ rad, $y_2 = \dot{\phi}$ rad/sec

D_i from Table 3 $y_3 = \dot{Z}$ m/sec

$y_4 = \beta$ rad, $y_5 = M_B$ m-kg

Table 5. Vehicle Response Amplitudes with Minimax Controller and Typical Winds

Interval	Cost Variable		Synthetic Wind Profile $\alpha_w(48)$	$\alpha_w(56)$	$\alpha_w(64)$	$\alpha_w(72)$	$\alpha_w(80)$
t=36							
I_3	(ϕ rad)	10	0.17	0.09	0.04		
	($\dot{\phi}$ rad/sec)	10^2	0.20	0.13	0.10		
	($\dot{Z}$ m/sec)	10^{-1}	0.20	0.10	0.06		
	(β rad)	10	0.09	0.04	0.04		
	(M_B m-kg)	10^{-6}	1.70	1.10	0.57		
	(Z m)	10^{-2}	0.14	0.05	0.02		
t=52							
I_4	(ϕ rad)	10	0.40	0.34	0.16	0.06	
	($\dot{\phi}$ rad/sec)	10^2	0.33	0.45	0.21	0.15	
	($\dot{Z}$ m/sec)	10^{-1}	0.25	0.24	0.12	0.07	
	(β rad)	10	0.06	0.07	0.02	0.01	
	(M_B m-kg)	10^{-6}	1.70	2.36	1.06	0.69	
	(Z m)	10^{-2}	0.33	0.20	0.09	0.02	
t=60							
I_5	(ϕ rad)	10	0.74	0.81	0.55	0.25	0.00
	($\dot{\phi}$ rad/sec)	10^2	0.50	0.68	0.66	0.30	0.07
	($\dot{Z}$ m/sec)	10^{-1}	0.36	0.24	0.18	0.10	0.02
	(β rad)	10	0.02	0.02	0.03	0.03	0.03
	(M_B m-kg)	10^{-6}	1.73	1.85	2.51	1.18	0.64
	(Z m)	10^{-2}	0.40	0.31	0.21	0.09	0.00
t=68							
I_6	(ϕ rad)	10	0.83	0.96	0.89	0.58	0.14
	($\dot{\phi}$ rad/sec)	10^2	0.66	0.70	0.60	0.58	0.19
	($\dot{Z}$ m/sec)	10^{-1}	1.66	1.49	0.66	0.09	0.14
	(β rad)	10	0.11	0.11	0.07	0.13	0.07
	(M_B m-kg)	10^{-6}	0.56	0.74	2.02	3.16	1.13
	(Z m)	10^{-2}	0.41	0.31	0.23	0.13	0.07
t=76							
I_7	(ϕ rad)	10	0.78	0.92	0.87	0.58	0.35
	($\dot{\phi}$ rad/sec)	10^2	1.29	1.38	1.24	0.44	0.47
	($\dot{Z}$ m/sec)	10^{-1}	2.18	2.12	1.24	0.33	0.25
	(β rad)	10	0.11	0.15	0.14	0.20	0.26
	(M_B m-kg)	10^{-6}	0.59	1.35	1.26	2.45	2.71
	(Z m)	10^{-2}	2.00	1.84	0.77	0.13	0.23
t=84							