

I. 04

THERMAL MODELS OF JUPITER AND SATURN*

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ABSTRACT

Models corresponding to completely convective structure are calculated for Jupiter and Saturn. The equations of state used in the calculation are those given by DeMarcus (1958), as well as more recent results due to Salpeter and Zepolsky (1967). The models are chemically homogeneous and without a dense core. In the case of Jupiter, chemical abundances are chosen to be $X = 0.54$, $Y = 0.46$, in accordance with a recent observational determination (Beckman 1967). The central temperature as estimated from the surface condition is about $15,000^{\circ}\text{K}$. The model of Jupiter is in full accord with the observed gravitational moments. The temperature in the Jovian interior is estimated to be everywhere above the melting temperature of metallic hydrogen, such that the planet is probably entirely fluid.

Convective models of Saturn indicate that it has a central temperature of about $10,000^{\circ}\text{K}$, and is richer in helium than Jupiter. Thermal effects apparently play a much larger role in Saturn than in Jupiter. The Saturnian models do not provide good agreement with the observed gravitational moments, possibly owing to inadequacies in the assumed equation of state.

*Supported in part by the National Science Foundation [GP-7976] and the Office of Naval Research [Nonr-220(47)].



I. INTRODUCTION

In a previous paper (Hubbard 1968; referred to here as paper I), it was argued that the net flux of energy from Jupiter estimated by Low (1966) implies a superadiabatic temperature gradient in the interior, and therefore the planet is probably completely convective. This conclusion was based upon the model planets calculated by DeMarcus (1958; DM in the following), and Peebles (1964), together with some additional assumptions concerning the thermodynamics of metallic hydrogen at finite temperature, as well as a new result for the thermal conductivity of metallic hydrogen. The purpose of this paper is to calculate completely convective model planets for Jupiter and Saturn using the temperature effects on the equations of state as given in I, together with zero-temperature equations of state for hydrogen and helium as given by DM, or more recently calculated by Salpeter and Zepolsky (1967; SZ), and an adiabatic temperature distribution.

The assumption that Jupiter and Saturn are completely convective imposes certain constraints on possible models, the most important of these being that Jupiter and Saturn must be chemically homogeneous. In the case of Jupiter, using the observed luminosity, and the mixing length theory of convection, one can readily verify that Jupiter must have mixed completely over the age of the solar system, and this conclusion is probably also valid for Saturn (see §V). It must be conceded that if Jupiter or Saturn were provided with a dense core of matter with a sufficiently large thermal conductivity, such a core would be stable against convection and would thus be able to preserve its chemical identity. On the other hand, the previous history of the planet would have to be such that it would be possible for

the core to fractionate out at some time. As discussed by Öpik (1962) and Peebles (1964), diffusion under the influence of a gravitational field is altogether inadequate to provide such a separation. If the primeval Jupiter and Saturn were high temperature objects with extended radius, undergoing rapid convection, it is doubtful that a separation could ever occur, and in this paper we proceed under the assumption that Jupiter and Saturn possess no high density core differing in chemical composition from the remainder of the planet.

The models of Jupiter and Saturn calculated by DM are chemically similar to the Sun, while Peebles has found Jupiter and Saturn to have an even larger hydrogen to helium ratio than the Sun. On the other hand, direct observational estimates of abundances in the Jovian atmosphere tend to suggest that Jupiter has a smaller hydrogen to helium ratio than the Sun. Öpik (1962), upon reviewing the occultation data of Baum and Code (1953), has concluded that the Jovian atmosphere is essentially pure helium. The most recent estimate of Jovian abundances is by Beckman (1967), using an analysis based on the theory of pressure broadening of methane lines in a hydrogen-helium atmosphere. Beckman finds the fraction of hydrogen by mass, X , to be about 0.54, with the remaining fraction mostly helium. Other rather disparate values are to be found in the literature (e.g., Lasker 1963, Trafton 1967). A mechanism for the enrichment of the Jovian atmosphere in helium relative to the interior of the planet has been proposed by Smoluchowski (1967); in this discussion, however, we take the point of view that the atmospheric abundances are characteristic of Jupiter and Saturn as a whole. In the case of Jupiter, we take Beckman's result to be correct and assume $X = 0.54$, Y (helium abundance by mass) = 0.46, and assume that the minor constituents contribute negligibly to the equations of state.

In the case of Jupiter, it should be evident that the planetary structure is greatly overdetermined if we fix the chemical composition, dispense with a dense core of adjustable size, and require the temperature distribution in the interior to be consistent with the observed luminosity and age of the solar system, as well as requiring a fit to directly observed parameters such as mass, mean radius, and gravitational moments J and K .¹ Thus a fairly severe test of the assumed equations of state results.

In § II below, we discuss the DM and SZ equations of state, together with temperature modifications. In § III, we discuss the model atmospheres of Jupiter and Saturn as calculated by Trafton (1967), and consider how these atmospheres may be used to estimate the central temperature for a completely convective planet. In § IV are presented the results of the model planet calculations, which are discussed in § V.

II. EQUATIONS OF STATE

Two different procedures may be used to calculate the equation of state of the metallic phase of hydrogen at zero temperature. The Wigner-Seitz method, which was first employed by Wigner and Huntington (1935), replaces a unit cell of the lattice with a spherical volume and solves for the wave function of the electrons in the self-consistent field of the protons plus electrons subject to periodic boundary conditions at the surface of the unit cells. The Thomas-Fermi-Dirac method, in contrast, also uses a spherical unit cell and a self-consistent field, but the local electron density is

¹The assumed values for mass, mean radius, and gravitational moments of Jupiter and Saturn are the same as those given by Peebles (1964).

given by the local chemical potential according to Fermi statistics, rather than by the absolute square of the solution to the Schrödinger equation. In both methods, exchange and correlation corrections are included in an appropriate manner. Since hydrogen has only one electron per unit cell, the Wigner-Seitz method would seem to give superior results to the TFD method in this case. However, a comparison of model planets calculated according to both procedures is desirable in that it gives a quantitative indication of the basic uncertainty in the theoretical metallic hydrogen equation of state.

The TFD equation has been solved by SZ for a large number of elements, including hydrogen and helium, and their equation of state includes a more modern result for the correlation energy than was used by DM. In Figure 1, we plot the ratio of the SZ density to the DM density for a range of pressures, for hydrogen and helium. The convergence of the two results at high pressures is not accidental, since Salpeter (1961) has shown that the leading corrections to the ideal Fermi gas pressure in the limit of high pressure are the same whether calculated by Wigner-Seitz or TFD theory. The convergence of the DM and SZ equations of state takes place at such a low pressure that Jovian models should be reasonably insensitive to which equation of state is used. This rapid convergence is due essentially to the fact that the electron wave functions become nearly plane waves at densities greater than about 1 g cm^{-3} , which is equivalent to saying that the electron distribution becomes nearly uniform. A quantitative measure of the uniformity of the electron distribution is provided by considering the asymptotic form of the pressure given in SZ:

$$\frac{P}{P_0} = 1 - \left[\frac{3}{2} \left(\frac{4}{9\pi} \right)^{2/3} z^{2/3} + \frac{5}{4\pi} \left(\frac{4}{9\pi} \right)^{1/3} \right] r_e, \quad (1)$$

where P_0 is the pressure of an ideal Fermi gas of electrons at zero temperature, Z is the ionic charge in units of an electronic charge, and

$$\rho = 2.679 \mu_e / r_e^3, \quad (2)$$

where ρ is the density in g cm^{-3} and $\mu_e = 1$ (2) for hydrogen (helium). We then define an effective ionic charge Z_{eff} by requiring that the asymptotic pressure, formula (1), give the correct pressure even at low density when Z_{eff} is substituted for Z in the formula (see Fig. 2). Thus Z_{eff} can be interpreted as the ionic charge which gives the correct pressure if the electron distribution is assumed to be uniform and the electron density equal to the average electron density. Anticipating the results of the detailed model planet calculations, we find that for the center of Jupiter, Z_{eff} is about 0.8 for hydrogen and about 1.3 for helium. For Saturn's center, Z_{eff} is about 0.65 and 1.0 for hydrogen and helium respectively. The assumption of uniform electron density is therefore a reasonable approximation for the Jovian center, at least in the neighborhood of the protons. The accuracy of the SZ and DM equations of state is therefore probably much greater for the Jovian center than the Saturnian center.

Since the negative charge distribution at the Jovian center may at least in first approximation be regarded as uniform, it seems reasonable to apply the Monte Carlo calculations of Brush, Sahlin, and Teller (1966; BST) to obtain the thermodynamic functions of the proton-alpha particle plasma in the planetary interior. The BST data are applicable to a strongly coupled one-component plasma in a uniform neutralizing background, at temperatures above the Debye temperature. For reasons discussed in paper I, the temperature in Jupiter's metallic hydrogen core is probably above the Debye temperature, and the same conclusion is probably true for Saturn.

Unfortunately, the BST data cannot be straightforwardly applied to the Jovian interior since Jupiter probably contains of the order of 20 per cent helium ions by number, and the plasma is therefore a multicomponent one. Furthermore, the electron background is in no way uniform in the neighborhood of the alpha particles. In order to treat this problem, the following approximations have been made: We assume that the plasma coupling parameter can be defined for a multicomponent system as

$$\Gamma = e^2 (\sum_{\alpha} Z_{\alpha}^2 n_{\alpha}) / (a kT \sum_{\alpha} n_{\alpha}), \quad (3)$$

$$a = (4\pi \sum_{\alpha} n_{\alpha} / 3)^{-1/3}, \quad (4)$$

where n_{α} is the number density of ions of species α . The coupling parameter Γ is then used as the argument in the corrections to the ideal gas thermodynamics as given by BST. This approximation is rigorously correct in the Debye-Hückel limit ($\Gamma \ll 1$), and we assume it is approximately correct in the strong-coupling limit ($\Gamma \gg 1$) as well. The correction to the ideal gas entropy as a function of Γ can be calculated from the BST data, and is plotted in Figure 3. To obtain the total entropy relative to a known zero point, we simply add the coupling correction to the entropy of an ideal gas of protons and alpha particles. Contributions to the entropy due to excitations of higher electron states are assumed to be negligible. The total pressure in the limit of uniform electron distribution is then given by the sum of BST pressure, ideal Fermi gas pressure, exchange pressure, and correlation pressure. Only the BST pressure is assumed to be temperature dependent. The BST pressure can be further broken down into the Coulomb pressure [which is negative; this has been calculated by Salpeter (1961)], and a contribution due to thermal vibrations of the ions (positive). The

latter contribution is the only one which really depends strongly on temperature for $\Gamma \gtrsim 30$.

An alternative and equivalent point of view is to regard the planetary matter as a Debye solid with a Debye temperature $\Theta \propto \rho^{1/2}$ (see paper I). The temperature perturbations to the zero temperature pressure are then due to a gas of phonons, and above the Debye temperature one obtains for the pressure

$$P = \frac{3}{2} n kT + P_0 \quad (5)$$

where P_0 is the zero temperature pressure, consisting of Fermi pressure, exchange pressure, Coulomb pressure, and correlation pressure, and n is the ion number density. For densities so low that the electron distribution is non-uniform, P_0 is given, for example, by the solution to the TFD equation, together with appropriate correlation corrections. At such densities, we assume that $\Theta \propto \rho^{1/2}$ is also valid, such that equation (5) remains correct. For a multicomponent system, we replace n by $\sum_{\alpha} n_{\alpha}$.

In order to isolate the temperature perturbation as in equation (5), it is necessary to verify that the temperature variations in P_0 are negligible. The source of temperature dependence in P_0 would be: (a) a change in shape of the Coulomb well due to a weakening of the coupling, (b) temperature variations in the exchange pressure, (c) temperature variations in the Fermi pressure. Let us consider typical conditions in the Jovian core, such that $T \sim 10^4$ °K, $\rho \sim 4$ g cm⁻³. Here the density is sufficiently high for the BST data to be approximately valid, and we find that relation (5) is equivalent to the BST result to within about 3 per cent, indicating that the shape of the Coulomb well is essentially that of an unscreened charge within a Wigner-Seitz sphere, the same as at zero

temperature. Under these conditions, $\Gamma \sim 30$, corresponding to the liquid phase in the BST model. Since the perturbations to the Fermi pressure go as T^2 for low temperature, one readily finds these perturbations to be completely negligible under the assumed conditions. For non-zero temperature, we can write the ratio of the exchange pressure, P_x , to the Fermi pressure P_f , in the form

$$\frac{P_x}{P_f} = -C' \frac{\left[F_{1/2}(\psi) F_{-1/2}(\psi) - \int_0^{F_{1/2}(\psi)} dF_{1/2} F_{-1/2}(F_{1/2}) \right]}{F_{3/2}(\psi) T^{1/2}}, \quad (6)$$

where

$$F_n(\psi) = \int_0^{\infty} \frac{x^n dx}{1 + \exp(x - \psi)}, \quad (7)$$

ψ is the electron chemical potential relative to kT , and C' is a constant independent of temperature and density (DeWitt 1966). At zero temperature, equation (6) reduces to the result given by Salpeter (1961). We find that the value of P_x/P_f is only about one per cent less under the assumed conditions than it is at zero temperature. The perturbation due to the phonon excitation amounts to about 10 per cent of the total pressure under these conditions, and it is therefore a valid approximation to neglect temperature variations in all other contributions to the pressure.

For the case of a mixture of ionic species, the pressure in TFD theory is determined by requiring the total volume of the system under a given pressure to be equal to the sum of the volumes of each ionic species. In this way we obtain the implicit equation of state for a mixture of hydrogen and helium; thus

$$\frac{1}{\rho} = \frac{X}{\rho_H(P)} + \frac{Y}{\rho_{He}(P)}, \quad (8)$$

where $\rho_\alpha(P)$ is the density of component α at pressure P . This procedure for calculating the equation of state of a mixture was also used by DM and Peebles. In the present discussion, P_0 is calculated using equation (8), and the total pressure is obtained using equation (5).

In the outer parts of the planet the physical conditions become more obscure. As the density decreases, the electrons tend to cluster more and more about the ions, and eventually the metallic phase becomes unstable and undergoes a transition to the molecular phase. The phase transition in hydrogen has been studied by DM, who finds that at zero temperature the transition from metallic to molecular hydrogen occurs at a pressure of about 2 megabars. This corresponds to a density in the metallic phase of about 1 g cm^{-3} . However, it has been suggested by Alder (1960) that the actual phase transition at zero temperature probably takes place near a pressure of 20 megabars. Furthermore, it has not been considered whether the helium component of the gas undergoes a similar polymorphic phase transition, and for models with a large helium abundance this question becomes relevant. An additional complication is that at densities around unity and lower, the temperature effects on the equation of state are quite large and undoubtedly affect the critical density.

Since the situation in the outer layers of Jupiter and Saturn is quite obscure, we have adopted the following essentially ad hoc procedure for treating this region. In the center, we assume that an adiabat is given as in the Debye model by

$$T = C\rho^{1/2}, \quad (9)$$

where C is a constant determined from the surface condition (see § III). As discussed in paper I, equation (9) is valid also in the liquid metallic

region, according to the RST data. The pressure is then determined as a function of ρ only by inserting equation (9) in equation (5). We then assume that the polymorphic phase transition takes place at a metallic density ρ_0 , and require that the pressure in the molecular phase at this point be equal to the pressure in the metallic phase. To calculate the pressure in the molecular phase, we have used the DM equation of state for molecular hydrogen together with either the DM or SZ equation of state for helium. The total pressure is then calculated by assuming $T = Cp^n$, where C is the same constant as in the core, and using equation (5), which now has no physical significance but is simply an interpolation relation. On physical grounds, one would expect $n \sim 0.4$ near the surface, since this is the adiabatic relation which one obtains for a molecular hydrogen gas at a temperature of around 200 $^{\circ}\text{K}$. We have calculated models with $n = 0.5$ and $n = 0.4$ in the envelope, and the results are quite insensitive to which value is used. Since there is a discontinuity in density at the polymorphic phase transition, an unrealistic discontinuity in temperature results at this point, but the discontinuity is quite small and has a negligible effect on the models.

The properties of the interpolation formula can be illustrated by checking its fit in the atmosphere. Let us consider typical atmospheric conditions for Jupiter, such that $T = 150$ $^{\circ}\text{K}$, $P = 1$ bar, $X = 0.54$. The corresponding density is then about 2.1×10^{-4} g cm^{-3} . The temperature given by $T = Cp^n$ is about 100 $^{\circ}\text{K}$ for $C = 6800$, $n = 0.5$, and about 230 $^{\circ}\text{K}$ for $C = 6800$, $n = 0.4$. The pressure given by the interpolation relation is thus 1.7 bar and 3.9 bar respectively. Therefore, the $n = 0.5$ interpolation relation gives a better fit to the surface temperature and pressure, although the $n = 0.4$ relation gives a better fit to the variation of pressure with

density near the surface.

In the models we have treated ρ_0 as an adjustable parameter, and have chosen it to conform to the estimates of Alder and DM respectively. Since the SZ equation of state for hydrogen is quite close to the DM equation of state for molecular hydrogen in the low density region, the models are particularly insensitive to the value of ρ_0 when the SZ equations of state are used for the core.

As in previous model planet calculations, we assume no phase transition occurs for helium and use the same equation of state (DM or SZ) in both core and envelope. As in paper I, we assume that the melting point of the Coulomb lattice in the metallic phase is given by $\Gamma = 40$.

III. THE SURFACE CONDITION

As shown by Hayashi and Nakano (1963), the structure of a completely convective low mass object can be estimated by calculating the entropy at the surface of the object, using a model atmosphere, and by requiring the surface entropy to equal the entropy in the deep interior where pressure ionization is presumably complete and the equations of state are known. The adiabat in the complex envelope is then guessed by smoothly interpolating between the surface and the interior. Such a procedure assumes, of course, that the zero point of the entropy at the surface and the interior is the same. The equation for the entropy per heavy particle of a mixture of hydrogen and helium at the surface and center of a low mass star are given by Hayashi and Nakano, and are employed here, with certain modifications. The expression used by Hayashi and Nakano to calculate the entropy of molecular hydrogen at the surface assumes that the temperature is sufficiently high

for the rotational modes to be treated classically, but this approximation is not valid for the Jovian or Saturnian atmosphere. For pure hydrogen, the rotational contribution to the entropy in the classical approximation is

$$\left(\frac{S}{k n_p}\right)_{\text{rot. class.}} = \frac{1}{2} \ln \left(\frac{I kT}{\hbar^2}\right) + \frac{1}{2} \quad (10)$$

where n_p is the number density of heavy particles (protons, in the case of pure hydrogen), and I is the moment of inertia of the H_2 molecule. The more precise quantum-mechanical expression is

$$\begin{aligned} \left(\frac{S}{k n_p}\right)_{\text{rot. Q.M.}} &= \\ &\frac{1}{2} \ln (g_g Z_g + g_u Z_u) \\ &+ \frac{1}{2} \frac{\hbar^2}{(2IkT)(g_g Z_g + g_u Z_u)} \times \\ &\left\{ g_g \sum_{K=0,2,\dots} [(2K+1)(K+1)K] \exp \left[-\frac{\hbar^2}{2IkT} K(K+1) \right] \right. \\ &\left. + g_u \sum_{K=1,3,\dots} [(2K+1)(K+1)K] \exp \left[-\frac{\hbar^2}{2IkT} K(K+1) \right] \right\} \end{aligned} \quad (11)$$

(Landau and Lifshitz 1958). Here $g_g = 1/4$, $g_u = 3/4$, and

$$Z_g = \sum_{K=0,2,\dots} (2K+1) \exp \left[-\frac{\hbar^2}{2IkT} K(K+1) \right], \quad (12)$$

$$Z_u = \sum_{K=1,3,\dots} (2K+1) \exp \left[-\frac{\hbar^2}{2IkT} K(K+1) \right]. \quad (13)$$

The formula for the surface entropy given by Hayashi and Nakano can then be used, with the addition of a correction term equal to

$$\left(\frac{S}{k n_p}\right)_{\text{rot., Q.M.}} - \left(\frac{S}{k n_p}\right)_{\text{rot., class.}}$$

Contributions due to ammonia, methane, and other minor constituents are not included.

For the center of Jupiter, we use the expression given by Hayashi and Nakano for the entropy of a degenerate electron gas plus an ideal gas of protons and alpha particles, plus the correction term taken from BST (Fig. 1). Thus, we calculate the value of Γ at the center of the planet which gives agreement with the surface entropy, which then gives the central temperature and therefore the value of C . We have calculated C for three of Trafton's Jovian atmospheres and one Saturnian atmosphere, which are summarized as follows:

Jupiter model 1:

$$X = 1.0, T_e = 130 \text{ }^\circ\text{K}, T_c = 149 \text{ }^\circ\text{K}, P_c = 0.65 \text{ bar};$$

Jupiter model 2:

$$X = 1.0, T_e = 120 \text{ }^\circ\text{K}, T_c = 139 \text{ }^\circ\text{K}, P_c = 0.70 \text{ bar};$$

Jupiter model 3:

$$X = 0.2, T_e = 120 \text{ }^\circ\text{K}, T_c = 158 \text{ }^\circ\text{K}, P_c = 1.77 \text{ bar};$$

Saturn model 1:

$$X = 1.0, T_e = 100 \text{ }^\circ\text{K}, T_c = 118 \text{ }^\circ\text{K}, P_c = 0.50 \text{ bar}.$$

Here T_e is the effective temperature, and T_c and P_c are the temperature and pressure at the top of the convective zone. The convective zone starts

in all cases at a sufficiently large mean optical depth for the temperature gradient to be essentially equal to the adiabatic gradient. We then obtain the following results for C: Jupiter model 1, $C = 7800$; Jupiter model 2, $C = 7800$; Jupiter model 3, $C = 8500$; Saturn model 1, $C = 7500$. The value of C is apparently insensitive to the effective temperature, and can probably be safely extrapolated up to $T_e \sim 150$ °K. In all cases we have assumed $Z = 1$ for hydrogen, i.e., the protons are completely unscreened by the electrons. In model 3, $Z = 2$ was assumed for helium. If we assume that Γ can be computed using $Z = 1$ for helium and hydrogen in model 3, C is reduced to 5300. In general, the value of C is reduced due to the effect of electron screening in reducing the value of Z. The reason for this is that for a given value of Γ , the temperature must be lower if Z is lower.

We have adopted the following "observationally determined" values for Jupiter: $X \cong 0.54$, $T_e \cong 150$ °K, $C \sim 7000$. In the case of Saturn, we assume only that $C \sim 7000$ and determine the remaining parameters from the models.

IV. THE MODEL PLANETS

Once the value of X and C is prescribed, the equation of hydrostatic equilibrium can be integrated to give the mass, radius, and density run for a model planet. In the present discussion, the central density was guessed and the equation of hydrostatic equilibrium integrated by a centered-difference implicit scheme out to a radius where perfect gas conditions obtained, and then integrated analytically to the surface. About 35 mesh points were used, with a spacing of about 4×10^8 cm in the core and a higher density near the surface. Several models were calculated with 100 mesh points, but very little increase in numerical accuracy resulted. The models

were iterated to give the correct mass to within one per cent, and then X or C was adjusted to obtain the correct mean radius to within one per cent. The gravitational moments J and K were then calculated according to the prescription given by Peebles (1964). A first-order correction to the equation of hydrostatic equilibrium due to rotation was also included, following Peebles.

It should be noted that the convective Jovian model calculated by Peebles has essentially a C of about 2000; thus Peebles' Jupiter model is much colder than the one considered here. Since the Jovian model considered here is more thermally expanded, it is possible to make it richer in helium and still obtain the correct radius. The crucial test is whether the correct values for J and K are obtained.

A further check on the Jovian models is provided by considering the energy budget. Since the pressure is insensitive to the temperature through most of the planet, the energy radiated is derived mainly from the thermal energy of the ions, which may be estimated to be about 3 kT per heavy particle (see paper I). The total thermal energy is then obtained by integrating this over the mass of the planet, using $T = Cp^{1/2}$. We then use Low's estimate for the net flux of energy from Jupiter, $H \sim 2 \times 10^4$ erg/cm² sec to obtain the luminosity. The time scale which results from dividing the total thermal energy by the luminosity should be greater than or of the order of the age of the solar system, $\sim 5 \times 10^9$ years.

The first group of Jovian models, which were calculated using the DM equation of state, were not entirely satisfactory since the gravitational moments were about 3 to 5 per cent too small. It would have been possible, however, to force agreement in J and K by increasing X to about 0.61 and reducing C to about 6400. This is probably not outside the range of uncertainty

in these quantities. Models J1 through J7 are tabulated in Table 1; these models indicate the relative change in various quantities as a function of alteration of uncertain parameters in the equations of state. In this table, ρ_c is the central density of the model in g cm^{-3} , τ is the time scale in 10^9 years, and n is the exponent which appears in the assumed temperature relation in the molecular envelope, $T = C\rho^n$. In most of the models we have assumed that the critical pressure for the polymorphic phase transition is about 5 megabars. For previously discussed reasons, the models calculated with the SZ equation of state are quite insensitive to the critical pressure. Model J5 was calculated on the assumption that the critical pressure is about 20 megabars. Even in this case, the sensitivity to the change is not strong.

In general, the models calculated with the SZ equations of state allowed a somewhat better fit to the observed parameters. Model J7, which we consider to be the best of the seven Jovian models, is tabulated in detail in Table 2. We emphasize that the temperatures given for the outer portions of the planet are given by the interpolation relation and are therefore only of qualitative significance. The quantity P_T/P is the fraction of the total pressure due to temperature effects. The coupling parameter Γ was calculated assuming $Z = 1$ for both hydrogen and helium. If we assume the ions form a lattice for $\Gamma > 40$, this model implies a completely liquid planet. Note that the coupling increases toward the surface, so that solidification will occur first in a surface crust as the cooling continues. The uncertainties in estimating the melting temperature are so great that it is entirely possible that Jupiter has a solid hydrogen layer somewhat below the surface, with a liquid metallic hydrogen core.

The cooling time given for all models has been calculated assuming a heat capacity of 3 k per heavy particle. However, for the range of Γ given in Table 3, a heat capacity of about 2.5 k is more appropriate, according to the data of BST. Upon making this correction, the cooling time for model J7 becomes 5.4×10^9 years.

In general, the Saturnian models calculated according to the procedures outlined above are rather unsatisfactory. Since the central density is so low, the interpolation relation is essentially applied to the entire planet and good results would be entirely fortuitous. For all models calculated, it was necessary to increase C to unrealistically large values and reduce X to about 0.2 to obtain good agreement with the observed gravitational moments. From the arguments given in § III, it seems likely that C is approximately the same as in Jupiter; however, the central density is so low that the BST data are probably not applicable even at the center. The effect of a large amount of helium is also uncertain since Trafton has only calculated a pure hydrogen atmosphere for Saturn. In any event, we consider it unlikely that the central temperature of Saturn could be larger than the central temperature of Jupiter.

The Saturnian models are summarized in Table 3. The SZ equations of state provided no significant improvement over the models calculated with the DM equations of state, a further indication that the difficulty lies in the interpolation relation. To illustrate the relative role of temperature in the case of Jupiter and Saturn, we plot P_T/P for both planets in Figures 4 and 5. In the case of Saturn, temperature effects on the pressure can hardly be regarded as a perturbation, even at the center.

An appropriate modification of the interpolation relation might produce more realistic results for the Saturnian gravitational movements, but such

a procedure would not be particularly illuminating. Instead, we content ourselves with drawing the following qualitative conclusions: (a) Saturn is probably richer in helium than Jupiter, (b) the central temperature of Saturn is of the order of $10,000^{\circ}\text{K}$. These conclusions are, of course, based upon the assumption that Saturn is convective.

V. CONCLUSIONS

The convective model of Jupiter presented here is fully consistent with the conclusions reached in paper I. The weakness of the model lies in the treatment of the outermost layers, where the matter is probably in a dense fluid phase of molecular hydrogen, and where temperature perturbations to the pressure are not negligible. The problem is further complicated by the possibility of a liquid-solid phase transition in the metallic hydrogen near the surface, which may affect the possibility of convection in this region. The treatment of this region of the planet may ultimately depend upon experimental data at pressures not presently attainable.

How well is the chemical composition of Jupiter determined theoretically? If we increase the hydrogen abundance to about 80 per cent by weight, decreasing the internal temperature simultaneously to obtain the proper radius, the gravitational moments become too large. However, they can be brought into agreement with the observed values by postulating a small high density core; thus we recover Peebles' model of Jupiter, particularly if the DM equations of state are employed. However, Peebles' model has such a low value of C that it is probably not consistent with Trafton's calculation of the surface condition, assuming complete convection. Furthermore, if Jupiter's intrinsic luminosity is as large as the value adopted in this paper, the Peebles model would not possess sufficient energy to radiate over

the age of the solar system. Therefore, a better observational value for Jupiter's luminosity may help to discriminate between Peebles' model with a large hydrogen abundance and the present model (J7) with a hydrogen abundance somewhat smaller than the solar hydrogen abundance. An improved determination of atmospheric abundances is evidently also desirable.

The model planets calculated for Saturn cannot be said to provide much support for the possibility that it is convective. Öpik (1962) has speculated that Saturn may possess a net luminosity approximately one third of Jupiter's, and Low (1966) has mentioned observational evidence which does not contradict this view. The model planets presented here are consistent with this hypothesis, for if we estimate the thermal energy in Saturn from the models and divide by the time scale of the solar system, 5×10^9 years, the predicted luminosity is from one third to one fifth that of Jupiter, depending on the value chosen for the heat capacity of the Saturnian model. Using the same arguments as in paper I, it can be shown that such a luminosity would be sufficient to support convection in the metallic hydrogen core at least. If Alder's surmise is correct, and the critical pressure for the transition from molecular hydrogen to metallic hydrogen is about 20 megabars, then metallic hydrogen probably does not exist in Saturn at all. Clearly, a good observational determination of the Saturnian luminosity and atmospheric chemical composition, and a direct measurement of the existence or nonexistence of a Saturnian magnetic field would provide much information on the state of the Saturnian interior.

It is a pleasure to thank Drs. J. N. Bahcall, E. J. Öpik, and H. S. Zepolsky for helpful discussions.

All numerical work was performed at the Booth Computing Center of the California Institute of Technology.

TABLE 1
JOVIAN MODELS

Number	X	C	ρ_c	τ	J	K	Equation of State
Observed	0.54	7000	--	5	0.0221	0.0025	--
J1	0.59	7000	4.24	7.4	0.0213	0.0023	DM, n = 0.5
J2	0.60	6700	4.20	7.1	0.0214	0.0024	DM, n = 0.5
J3	0.59	7000	4.23	7.2	0.0211	0.0023	DM, n = 0.4
J4	0.60	6700	4.20	6.9	0.0216	0.0023	DM, n = 0.4
J5	0.59	7000	4.19	6.5	0.0210	0.0023	DM, n = 0.4*
J6	0.54	7000	4.42	6.9	0.0211	0.0023	SZ, n = 0.4
J7	0.56	6800	4.23	6.5	0.0221	0.0026	SZ, n = 0.4

* $P_{CR} = 20 \text{ mb}$

TABLE 2
JUPITER MODEL (J7)

Radius ($\times 10^9$ cm)	ρ ($g\text{ cm}^{-3}$)	P (mb)	T ($^{\circ}\text{K}$)	P_T/P	Γ
0.0	4.23	36.6	14000	0.13	23
0.4	4.21	36.2	13950	0.13	23
0.8	4.14	35.1	13840	0.14	23
1.2	4.03	33.3	13660	0.14	23
1.6	3.88	30.9	13400	0.14	23
2.0	3.69	28.0	13070	0.14	24
2.4	3.47	24.7	12660	0.15	24
2.8	3.21	21.2	12190	0.15	24
3.2	2.93	17.7	11640	0.16	25
3.6	2.63	14.3	11030	0.17	25
4.0	2.32	11.1	10360	0.18	26
4.4	2.00	8.2	9620	0.19	26
4.8	1.68	5.7	8500	0.21	28
5.4	1.17	2.9	7200	0.24	
6.0	0.75	1.1	6000	0.33	
6.6	0.34	0.19	4400	0.65	
6.72	0.24	0.095	3800	0.81	
6.84	0.14	0.037	3100	0.94	
6.96	0.05	0.009	2000	0.99	

TABLE 3
SATURNIAN MODELS

Number	X	C	ρ_c	J	K	Equation of State
Observed	--	7000?	--	0.0250	0.0039	--
S1	0.38	7200	2.33	0.0325	0.0062	DM, n = 0.5
S2	0.36	8000	2.38	0.0313	0.0058	DM, n = 0.5
S3	0.28	8900	2.34	0.0317	0.0059	SZ, n = 0.4
S4	0.27	9000	2.39	0.0309	0.0055	SZ, n = 0.4

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FIGURE CAPTIONS

- Fig. 1. The ratio of density as given by the SZ equation of state to the density as given by the DM equation of state, for hydrogen and helium.
- Fig. 2. The value of the ionic charge which gives the correct pressure for a uniform electron distribution, computed on the basis of the SZ equation of state.
- Fig. 3. Correction to the entropy due to particle interactions, ΔS , calculated from the data of BST.
- Fig. 4. Density and temperature perturbation run for model J4.
- Fig. 5. Density and temperature perturbation run for model S2.

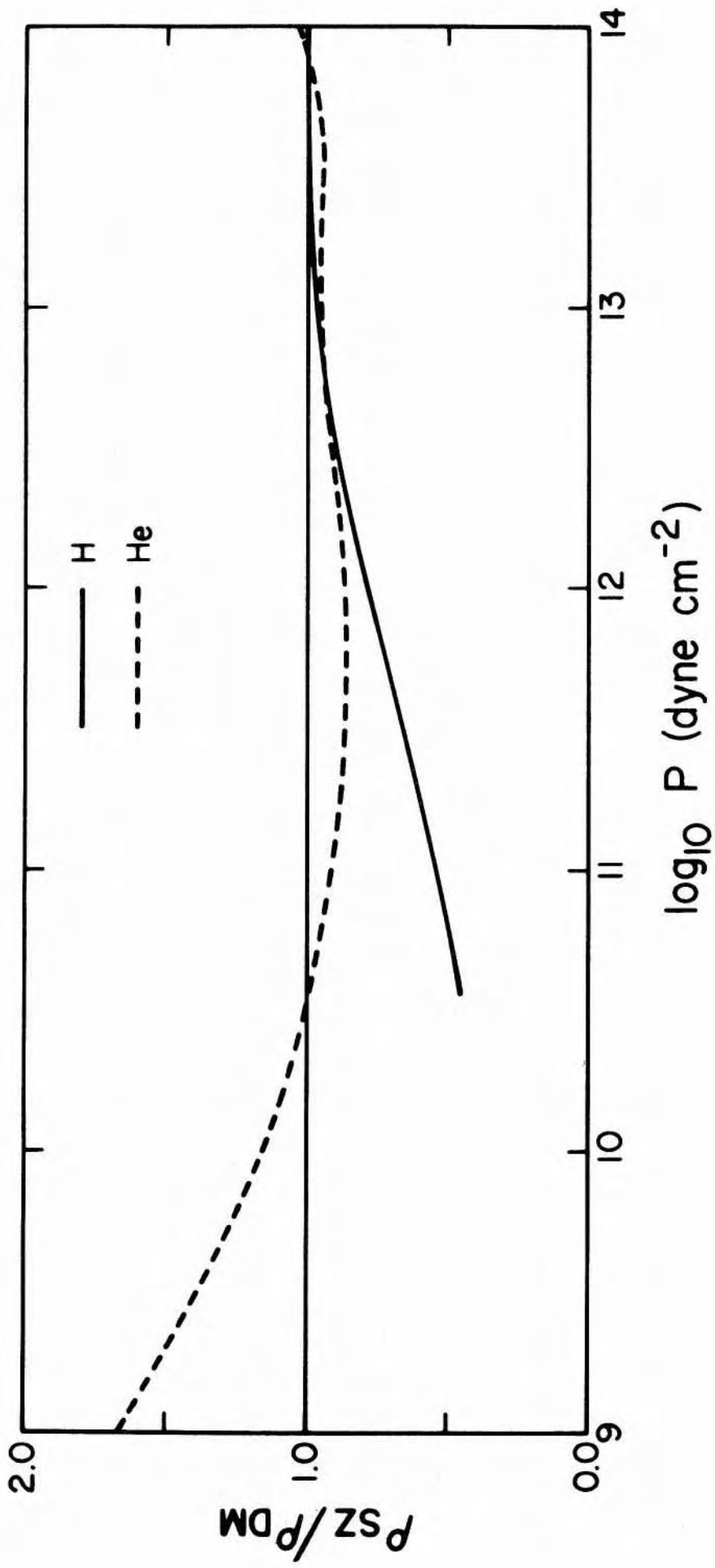


Fig. 1

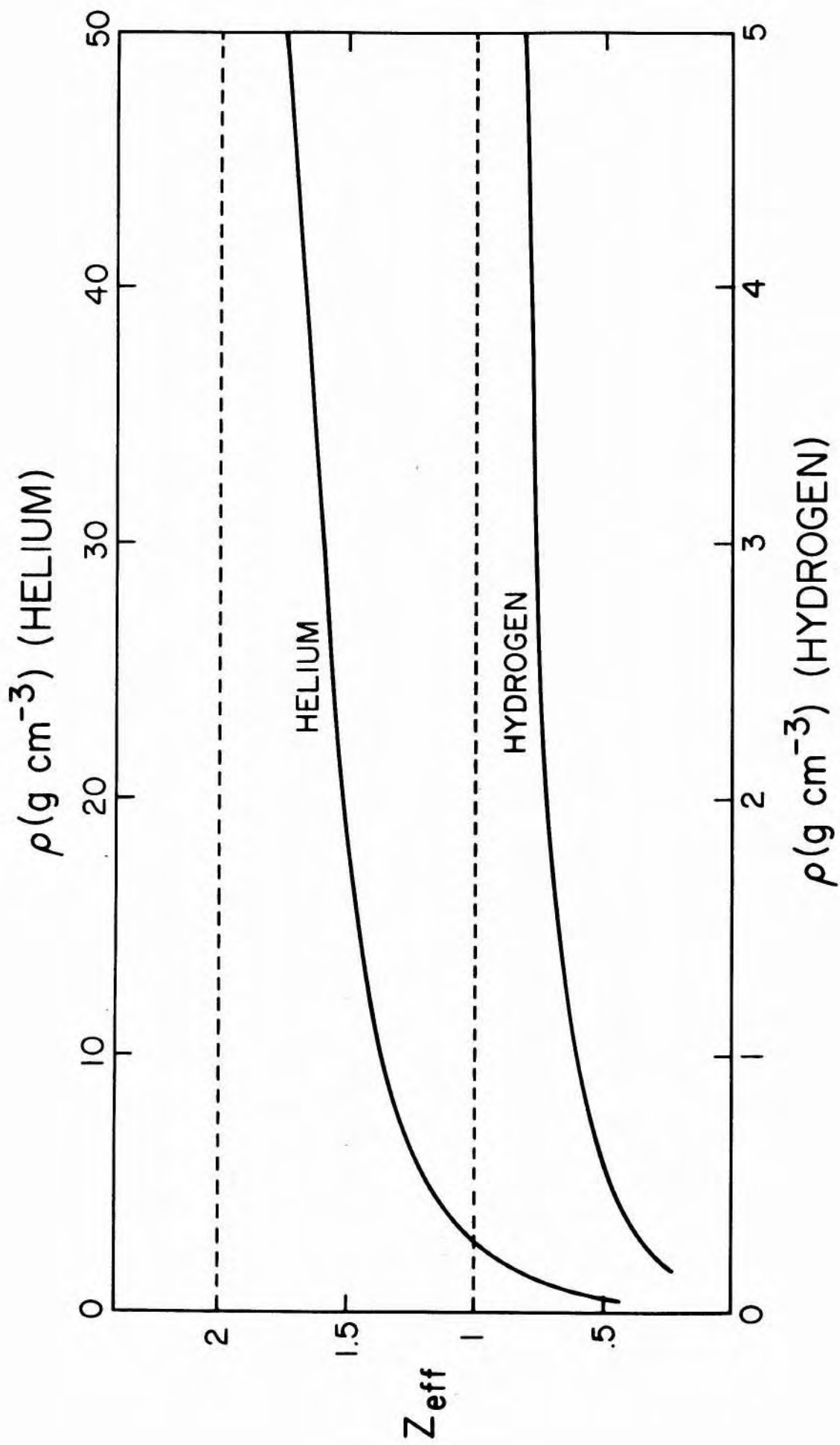


FIG. 2

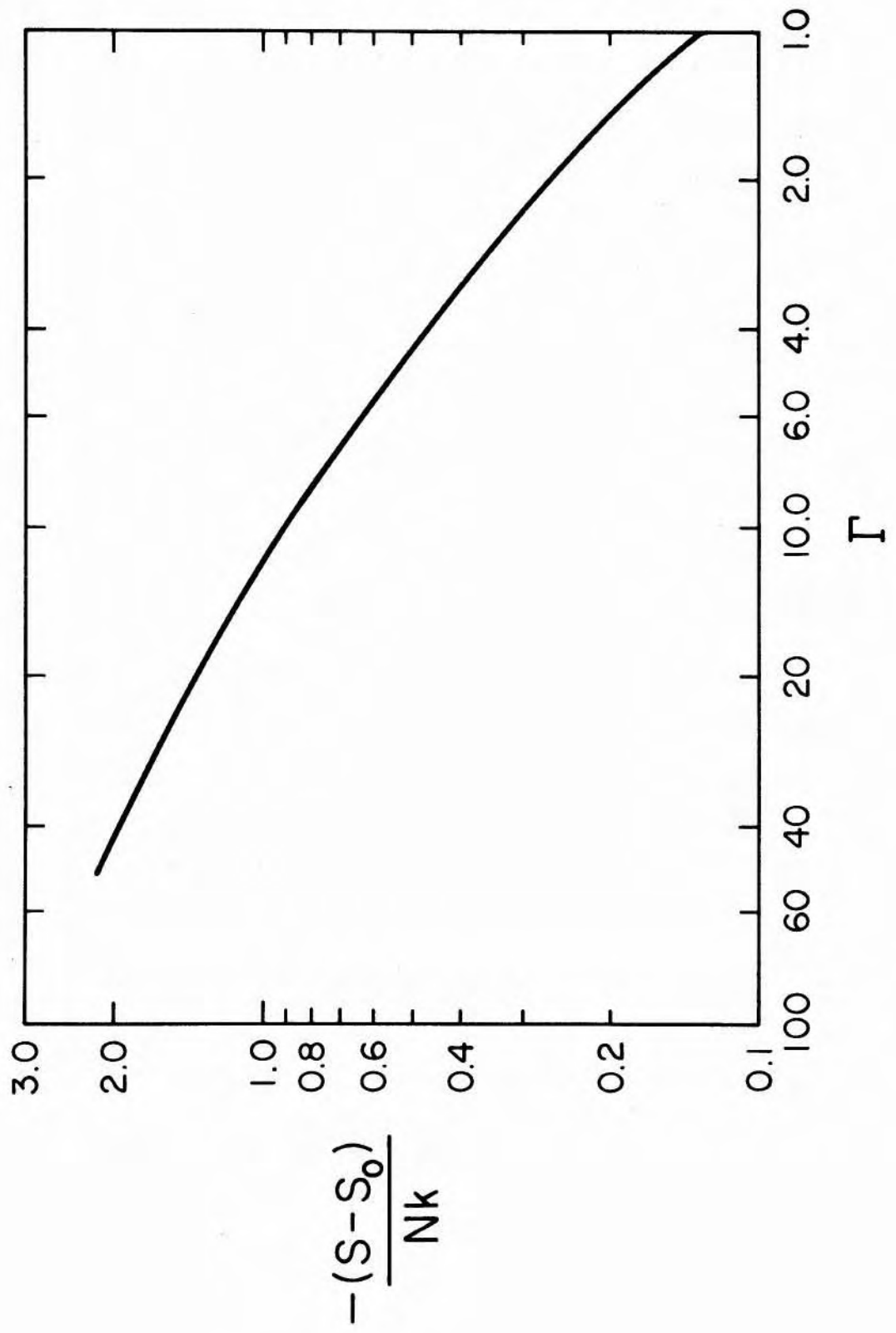


Fig. 3

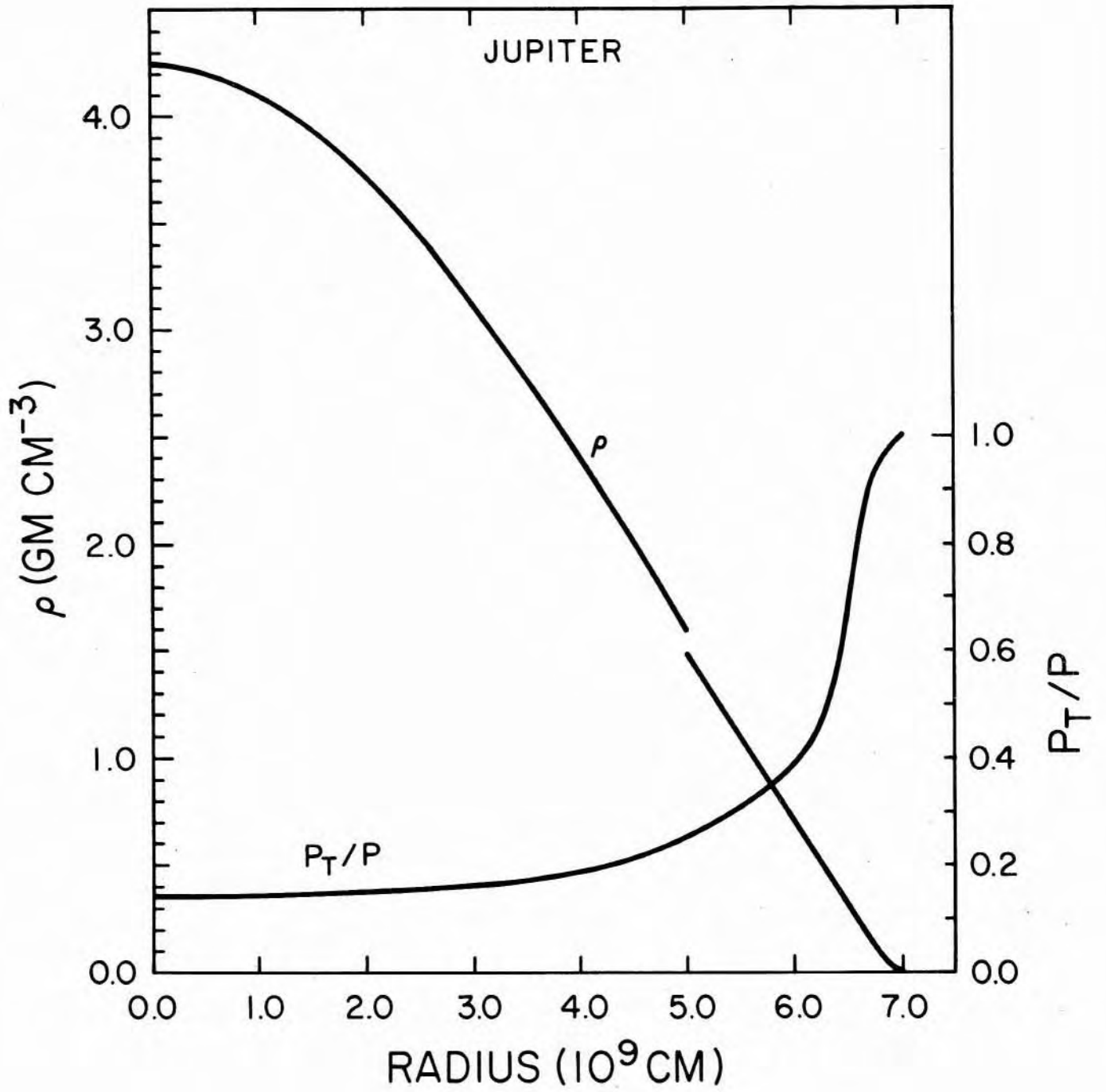


Fig. 4

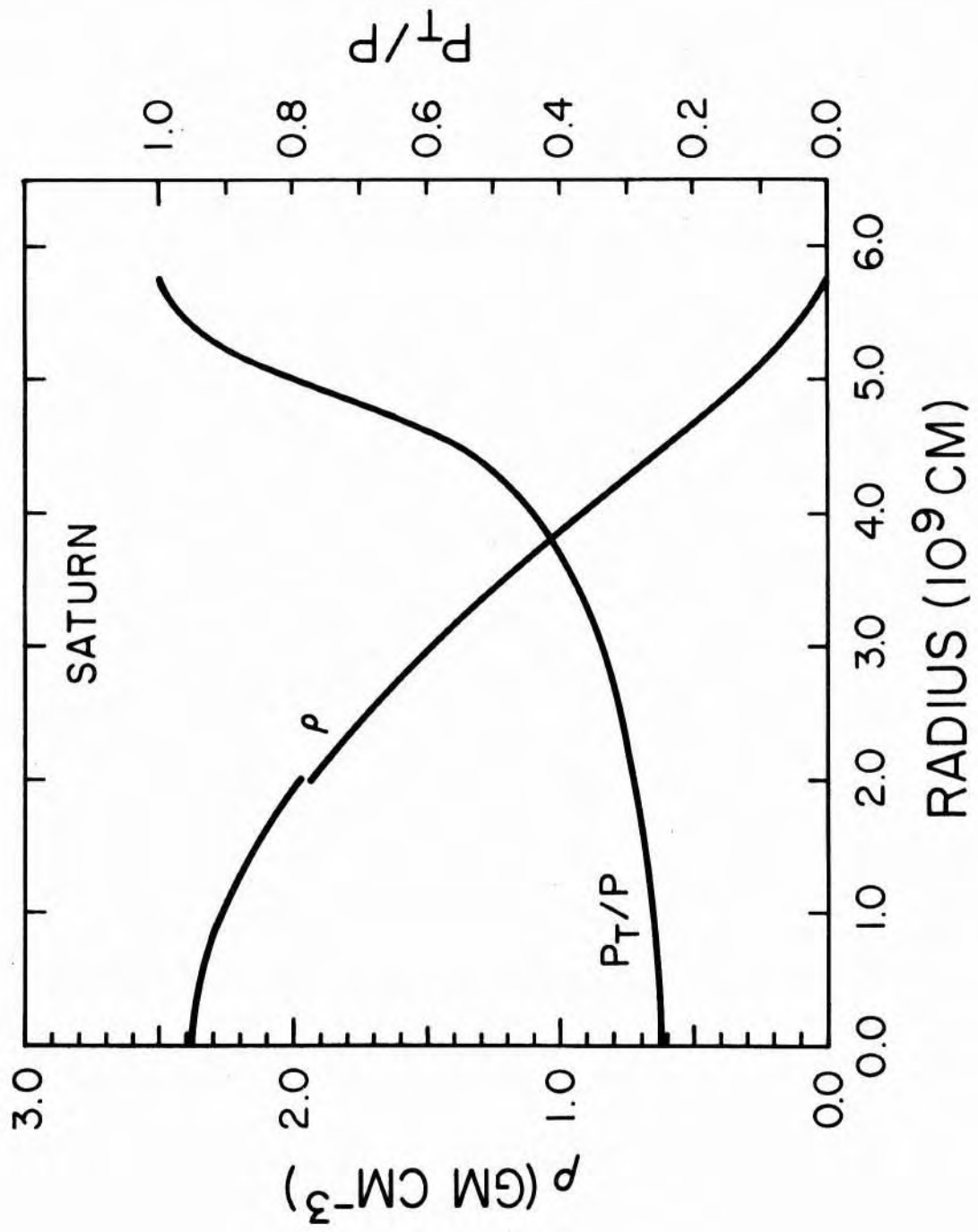


Fig. 5